# Discounted Cash Flow Valuation

## OPENING CASE

What do Chris Iannetta, John Lackey, and Matt Holliday have in common? All three are star athletes who signed big-money contracts during late 2009 or early 2010. Their contract values were reported as $8.35 million, $82.5 million, and $120 million, respectively. But reported numbers can be misleading. For example, catcher Chris Ianetta re-signed with the Colorado Rockies. His deal called for salaries of $1.75 million, $2.55 million, and $3.55 million over the next three years, respectively, with a contract buyout of $500,000 or a salary of $5,000,000 in four years. Not bad, especially for someone who makes a living using the "tools of ignorance" (jock jargon for a catcher's equipment).

A closer look at the numbers shows that Chris, John, and Matt did pretty well, but nothing like the quoted figures. Using Matt's contract as an example, the value was reported to be $120 million, but it was actually payable over several years. The terms called for a salary of $17 million per year for seven years, then a club option for $17 million in 2017 or a club buyout of $1 million. However, of the $17 million annual salary, $2 million each year was to be deferred and paid annually from 2020 to 2029. Since the payments are spread out over time, we must consider the time value of money, which means his contract was worth less than reported. How much did he really get? This chapter gives you the "tools of knowledge" to answer this question.

## 4.1 VALUATION: THE ONE-PERIOD CASE

Keith Vaughan is trying to sell a piece of raw land in Alaska. Yesterday, he was offered $10,000 for the property. He was about ready to accept the offer when another individual offered him $11,424. However, the second offer was to be paid a year from now. Keith has satisfied himself that both buyers are honest and financially solvent, so he has no fear that the offer he selects will fall through. These two offers are pictured as cash flows in Figure 4.1. Which offer should Mr. Vaughan choose?
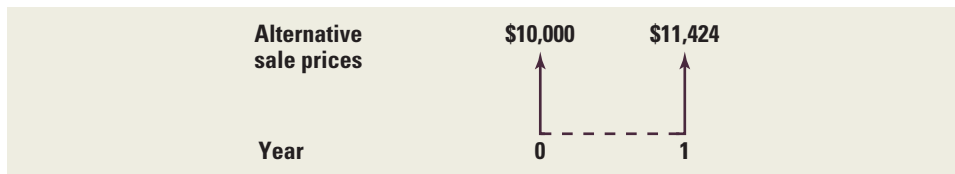
**FIGURE 4.1**
Cash Flow for
Mr. Vaughan's Sale

Mike Tuttle, Keith's financial adviser, points out that if Keith takes the first offer, he could invest the $10,000 in the bank at an insured rate of 12 percent. At the end of one year, he would have

$10,000 + (0.12 × $10,000) = $10,000 × 1.12 = $11,200
**Return of**　　　**Interest**
**principal**

Because this is less than the $11,424 Keith could receive from the second offer, Mr. Tuttle recommends that he take the latter. This analysis uses the concept of **future value** or **compound value**, which is the value of a sum after investing over one or more periods. The compound or future value of $10,000 at 12 percent is $11,200.

An alternative method employs the concept of **present value**. One can determine present value by asking the following question: How much money must Keith put in the bank today at 12 percent so that he will have $11,424 next year? We can write this algebraically as

PV × 1.12 = $11,424

We want to solve for present value (PV), the amount of money that yields $11,424 if invested at 12 percent today. Solving for PV, we have

$$PV = \frac{\$11,424}{1.12} = \$10,200$$

The formula for PV can be written as

**Present Value of Investment:**
$$PV = \frac{C_1}{1 + r} \qquad [4.1]$$

where $C_1$ is cash flow at date 1 and $r$ is the rate of return that Keith Vaughan requires on his land sale. It is sometimes referred to as the *discount rate*.

*Present value analysis* tells us that a payment of $11,424 to be received next year has a present value of $10,200 today. In other words, at a 12-percent interest rate, Mr. Vaughan is indifferent between $10,200 today or $11,424 next year. If you gave him $10,200 today, he could put it in the bank and receive $11,424 next year.

Because the second offer has a present value of $10,200, whereas the first offer is for only $10,000, present value analysis also indicates that Mr. Vaughan should take the second offer. In other words, both future value analysis and present value analysis lead to the same decision. As it turns out, present value analysis and future value analysis must always lead to the same decision.

As simple as this example is, it contains the basic principles that we will be working with over the next few chapters. We now use another example to develop the concept of net present value.

EXAMPLE 4.1

## Present Value

Lida Jennings, a financial analyst at Kaufman & Broad, a leading real estate firm, is thinking about recommending that Kaufman & Broad invest in a piece of land that costs $85,000. She is certain that next year the land will be worth $91,000, a sure $6,000 gain. Given that the guaranteed interest rate in the bank is 10 percent, should Kaufman & Broad undertake the investment in land? Ms. Jennings's choice is described in Figure 4.2 with the cash flow time chart.

### FIGURE 4.2

Cash Flows for Land Investment
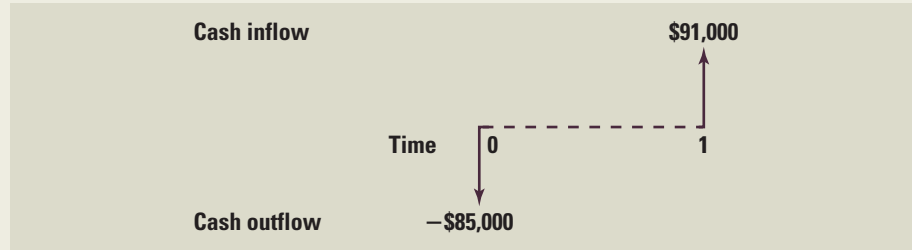
A moment's thought should be all it takes to convince her that this is not an attractive business deal. By investing $85,000 in the land, she will have $91,000 available next year. Suppose, instead, that Kaufman & Broad puts the same $85,000 into the bank. At the interest rate of 10 percent, this $85,000 would grow to

$(1 + .10) \times \$85,000 = \$93,500$

next year.

It would be foolish to buy the land when investing the same $85,000 in the financial market would produce an extra $2,500 (that is, $93,500 from the bank minus $91,000 from the land investment). This is a future value calculation.

Alternatively, she could calculate the present value of the sale price next year as

$\text{Present value} = \dfrac{\$91,000}{1.10} = \$82,727.27$

Because the present value of next year's sales price is less than this year's purchase price of $85,000, present value analysis also indicates that she should not recommend purchasing the property.

Frequently, businesspeople want to determine the exact *cost* or *benefit* of a decision. The decision to buy this year and sell next year can be evaluated as

**Net Present Value of Investment:**

$$-\$2,273 \quad = \quad -\$85,000 \quad + \quad \frac{\$91,000}{1.10}$$

|  | Cost of land today |  | Present value of next year's sales price |
|---|---|---|---|

The formula for NPV can be written as

$$\textbf{NPV} \quad = \quad -\textbf{Cost} \quad + \quad \textbf{PV} \qquad\qquad [4.2]$$

Equation 4.2 says that the value of the investment is −$2,273, after stating all the benefits and all the costs as of date 0. We say that −$2,273 is the **net present value (NPV)** of the investment. That is, NPV is the present value of future cash flows minus the present value of the cost of the investment. Because the net present value is negative, Lida Jennings should not recommend purchasing the land.

Both the Vaughan and the Jennings examples deal with perfect certainty. That is, Keith Vaughan knows with perfect certainty that he could sell his land for $11,424 next year. Similarly, Lida Jennings knows with perfect certainty that Kaufman & Broad could receive $91,000 for selling its land. Unfortunately, businesspeople frequently do not know future cash flows. This uncertainty is treated in the next example.

## Uncertainty and Valuation

Professional Artworks, Inc., is a firm that speculates in modern paintings. The manager is thinking of buying an original Picasso for $400,000 with the intention of selling it at the end of one year. The manager expects that the painting will be worth $480,000 in one year. The relevant cash flows are depicted in Figure 4.3.

**FIGURE 4.3**

Cash Flows for Investment in Painting
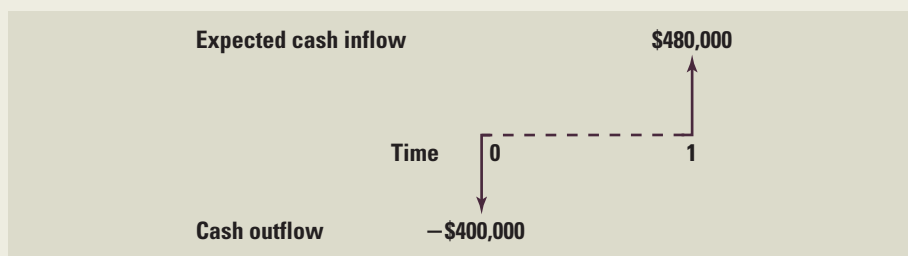


Of course, this is only an expectation—the painting could be worth more or less than $480,000. Suppose the guaranteed interest rate granted by banks is 10 percent. Should the firm purchase the piece of art?

Our first thought might be to discount at the interest rate, yielding

$$\frac{\$480,000}{1.10} = \$436,364$$

Because $436,364 is greater than $400,000, it looks at first glance as if the painting should be purchased. However, 10 percent is the return one can earn on a riskless investment. Because the painting is quite risky, a higher *discount rate* is called for. The manager chooses a rate of 25 percent to reflect this risk. In other words, he argues that a 25 percent expected return is fair compensation for an investment as risky as this painting.

The present value of the painting becomes

$$\frac{\$480,000}{1.25} = \$384,000$$

Thus, the manager believes that the painting is currently overpriced at $400,000 and does not make the purchase.

The preceding analysis is typical of decision making in today's corporations, though real-world examples are, of course, much more complex. Unfortunately, any example with risk poses a problem not faced by a riskless example. In an example with riskless cash flows, the appropriate interest rate can be determined by simply checking with a few banks. The selection of the discount rate for a risky investment is quite a difficult task. We simply don't know at this point whether the discount rate on the painting should be 11 percent, 25 percent, 52 percent, or some other percentage.

Because the choice of a discount rate is so difficult, we merely wanted to broach the subject here. We must wait until the specific material on risk and return is covered in later chapters before a risk-adjusted analysis can be presented.

## 4.2 THE MULTIPERIOD CASE

The previous section presented the calculation of future value and present value for one period only. We will now perform the calculations for the multiperiod case.

### Future Value and Compounding

Suppose an individual were to make a loan of $1. At the end of the first year, the borrower would owe the lender the principal amount of $1 plus the interest on the loan at the interest rate of $r$. For the specific case where the interest rate is, say, 9 percent, the borrower owes the lender

$$\$1 \times (1 + r) = \$1 \times 1.09 = \$1.09$$

At the end of the year, though, the lender has two choices. She can either take the $1.09—or, more generally, $(1 + r)$—out of the financial market, or she can leave it in and lend it again for a second year. The process of leaving the money in the financial market and lending it for another year is called **compounding**.

Suppose that the lender decides to compound her loan for another year. She does this by taking the proceeds from her first one-year loan, $1.09, and lending this amount for the next year. At the end of next year, then, the borrower will owe her

$$\$1 \times (1 + r) \times (1 + r) = \$1 \times (1 + r)^2 = 1 + 2r + r^2$$
$$\$1 \times (1.09) \times (1.09) = \$1 \times (1.09)^2 = \$1 + \$0.18 + \$0.0081 = \$1.1881$$

This is the total she will receive two years from now by compounding the loan.

In other words, the capital market enables the investor, by providing a ready opportunity for lending, to transform $1 today into $1.1881 at the end of two years. At the end of three years, the cash will be $1 \times (1.09)^3 = \$1.2950$.

The most important point to notice is that the total amount that the lender receives is not just the $1 that she lent out plus two years' worth of interest on $1:

$$2 \times r = 2 \times \$0.09 = \$0.18$$

The lender also gets back an amount $r^2$, which is the interest in the second year on the interest that was earned in the first year. The term, $2 \times r$, represents **simple interest** over the two years, and the term, $r^2$, is referred to as the *interest on interest*. In our example this latter amount is exactly

$$r^2 = (\$0.09)^2 = \$0.0081$$

When cash is invested at **compound interest**, each interest payment is reinvested. With simple interest, the interest is not reinvested. Benjamin Franklin's statement, "Money makes money and the money that money makes makes more money," is a colorful way of explaining compound interest. The difference between compound interest and simple interest is illustrated in Figure 4.4. In this example, the difference does not amount to much because the loan is for $1. If the loan were for $1 million, the lender would receive $1,188,100 in two years' time. Of this amount, $8,100 is interest on interest. The lesson is that those small numbers beyond the decimal point can add up to big dollar amounts when the transactions
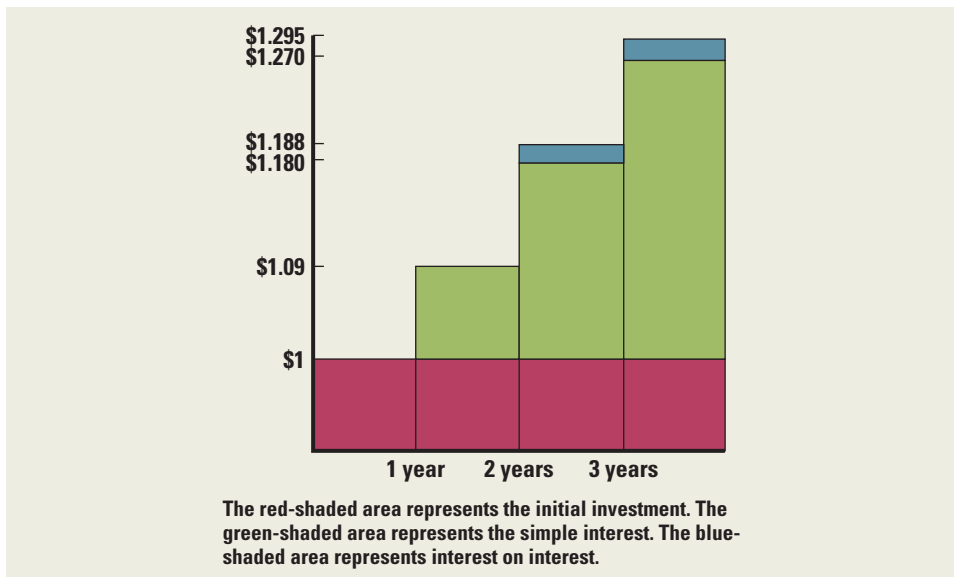
FIGURE 4.4

Simple and Compound
Interest



The red-shaded area represents the initial investment. The
green-shaded area represents the simple interest. The blue-
shaded area represents interest on interest.

are for big amounts. In addition, the longer-lasting the loan, the more important interest on
interest becomes.

The general formula for an investment over many periods can be written as

**Future Value of an Investment:**
$$FV = C_0 \times (1 + r)^T \tag{4.3}$$

where $C_0$ is the cash to be invested at date 0 (i.e., today), $r$ is the interest rate per period, and
$T$ is the number of periods over which the cash is invested.

**EXAMPLE 4.3**

### Interest on Interest

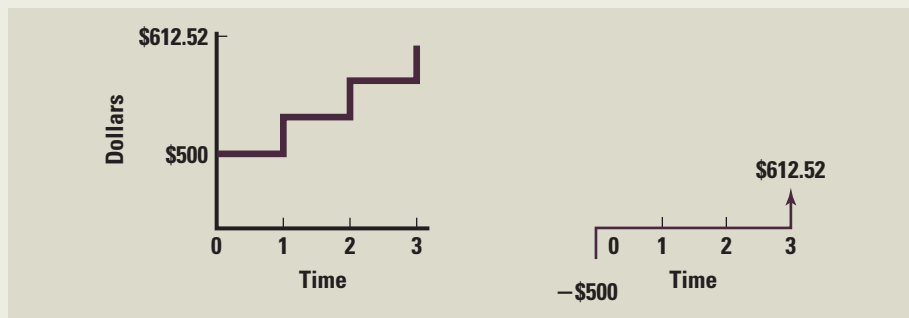Suh-Pyng Ku has put $500 in a savings account at the First National Bank of Kent. The account earns
7 percent, compounded annually. How much will Ms. Ku have at the end of three years?

$$\$500 \times 1.07 \times 1.07 \times 1.07 = \$500 \times (1.07)^3 = \$612.52$$

Figure 4.5 illustrates the growth of Ms. Ku's account.

**FIGURE 4.5**
Suh-Pyng Ku's Savings Account

EXAMPLE 4.4

## Compound Growth

Jay Ritter invested $1,000 in the stock of the SDH Company. The company pays a current dividend of $2, which is expected to grow by 20 percent per year for the next two years. What will the dividend of the SDH Company be after two years?

$$\$2 \times (1.20)^2 = \$2.88$$

Figure 4.6 illustrates the increasing value of SDH's dividends.

### FIGURE 4.6

The Growth of the SDH Dividends
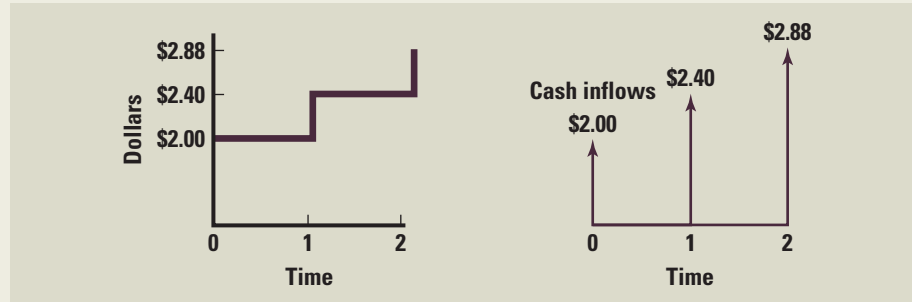


The two previous examples can be calculated in any one of four ways. The computations could be done by hand, by calculator, by spreadsheet, or with the help of a table. The appropriate table is Table A.3, which appears in the back of the text. This table presents *future value of $1 at the end of T periods*. The table is used by locating the appropriate interest rate on the horizontal axis and the appropriate number of periods on the vertical axis.

For example, Suh-Pyng Ku would look at the following portion of Table A.3:

| PERIOD | INTEREST RATE | | |
| --- | --- | --- | --- |
| | 6% | 7% | 8% |
| 1 | 1.0600 | 1.0700 | 1.0800 |
| 2 | 1.1236 | 1.1449 | 1.1664 |
| 3 | 1.1910 | 1.2250 | 1.2597 |
| 4 | 1.2625 | 1.3108 | 1.3605 |

She could calculate the future value of her $500 as

$$\underset{\text{Initial investment}}{\$500} \quad \times \quad \underset{\text{Future value of \$1}}{1.2250} \quad = \$612.50$$

In the example concerning Suh-Pyng Ku, we gave you both the initial investment and the interest rate and then asked you to calculate the future value. Alternatively, the interest rate could have been unknown, as shown in the following example.
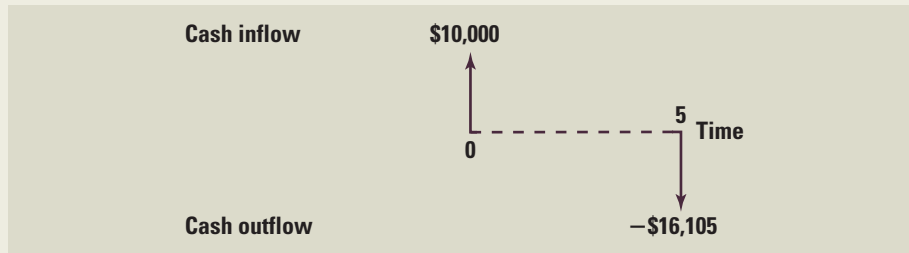
EXAMPLE 4.5

## Finding the Rate

Carl Voigt, who recently won $10,000 in the lottery, wants to buy a car in five years. Carl estimates that the car will cost $16,105 at that time. His cash flows are displayed in Figure 4.7.

What interest rate must he earn to be able to afford the car?

### FIGURE 4.7

Cash Flows for Purchase of Carl Voigt's Car



The ratio of purchase price to initial cash is

$$\frac{\$16,105}{\$10,000} = 1.6105$$

Thus, he must earn an interest rate that allows $1 to become $1.6105 in five years. Table A.3 tells us that an interest rate of 10 percent will allow him to purchase the car.

One can express the problem algebraically as

$$\$10,000 \times (1 \times r)^5 = \$16,105$$

where $r$ is the interest rate needed to purchase the car. Because $16,105/$10,000 = 1.6105, we have

$$(1 + r)^5 = 1.6105$$

Either the table or a hand calculator solves for $r$.

## The Power of Compounding: A Digression

Most people who have had any experience with compounding are impressed with its power over long periods of time. In fact, compound interest has been described as the "eighth wonder of the world" and "the most powerful force in the universe."[1] Take the stock market, for example. Ibbotson and Sinquefield have calculated what the stock market returned as a whole from 1926 through 2009.[2] They find that one dollar placed in these stocks at the beginning of 1926 would have been worth $2,591.82 at the end of 2009. This is 9.81 percent compounded annually for 84 years, i.e., $(\$1.0981)^{84} = \$2,591.82$, ignoring a small rounding error.

The example illustrates the great difference between compound and simple interest. At 9.81 percent, simple interest on $1 is 9.81 cents a year (i.e., $.0981). Simple interest over 84 years is $8.24 (84 × $.0981). That is, an individual withdrawing .0981 cents every year would have withdrawn $8.24 (84 × $.0981) over 84 years. This is quite a bit below the $2,591.82 that was obtained by reinvestment of all principal and interest.

The results are more impressive over even longer periods of time. A person with no experience in compounding might think that the value of $1 at the end of 168 years would be twice the value of $1 at the end of 84 years, if the yearly rate of return stayed the same.

[1] These quotes are often attributed to Albert Einstein (particularly the second one), but whether he really said either is not known. The first quote is also often attributed to Baron Rothschild, John Maynard Keynes, Benjamin Franklin, and others.

[2] *Stocks, Bonds, Bills and Inflation* [SBBI]. 2010 Yearbook, Ibbotson Associates, Chicago, 2010.

Actually the value of $1 at the end of 168 years would be the *square* of the value of $1 at the end of 84 years. That is, if the annual rate of return remained the same, a $1 investment in common stocks should be worth $6,717,530.91 [$1 × (2,591.82 × 2,591.82)].

A few years ago, an archaeologist unearthed a relic stating that Julius Caesar lent the Roman equivalent of one penny to someone. Since there was no record of the penny ever being repaid, the archaeologist wondered what the interest and principal would be if a descendant of Caesar tried to collect from a descendant of the borrower in the 20th century. The archaeologist felt that a rate of 6 percent might be appropriate. To his surprise, the principal and interest due after more than 2,000 years was vastly greater than the entire wealth on earth.

The power of compounding can explain why the parents of well-to-do families frequently bequeath wealth to their grandchildren rather than to their children. That is, they skip a generation. The parents would rather make the grandchildren very rich than make the children moderately rich. We have found that in these families the grandchildren have a more positive view of the power of compounding than do the children.

## EXAMPLE 4.6

### How Much for That Island?

Some people have said that it was the best real estate deal in history. Peter Minuit, director-general of New Netherlands, the Dutch West India Company's Colony in North America, in 1626 allegedly bought Manhattan Island from native Americans for 60 guilders' worth of trinkets. By 1667, the Dutch were forced to exchange it for Suriname with the British (perhaps the worst real estate deal ever). This sounds cheap, but did the Dutch really get the better end of the deal? It is reported that 60 guilders was worth about $24 at the prevailing exchange rate. If the native Americans had sold the trinkets at a fair market value and invested the $24 at 5 percent (tax free), it would now, about 384 years later, be worth about $3.3 billion. Today, Manhattan is undoubtedly worth more than $2.5 billion, and so, at a 5 percent rate of return, the native Americans got the worst of the deal. However, if invested at 10 percent, the amount of money they received would be worth about

$$\$24(1 + r)^T = 24 \times 1.1^{384} = \$188 \text{ quadrillion}$$

This is a lot of money. In fact, $188 quadrillion is more than all the real estate in the world is worth today. Note that no one in the history of the world has ever been able to find an investment yielding 10 percent every year for 384 years.

### Present Value and Discounting

We now know that an annual interest rate of 9 percent enables the investor to transform $1 today into $1.1881 two years from now. In addition, we would like to know:

> How much would an investor need to lend today so that she could receive $1 two years from today?

Algebraically, we can write this as

$$PV \times (1.09)^2 = \$1$$

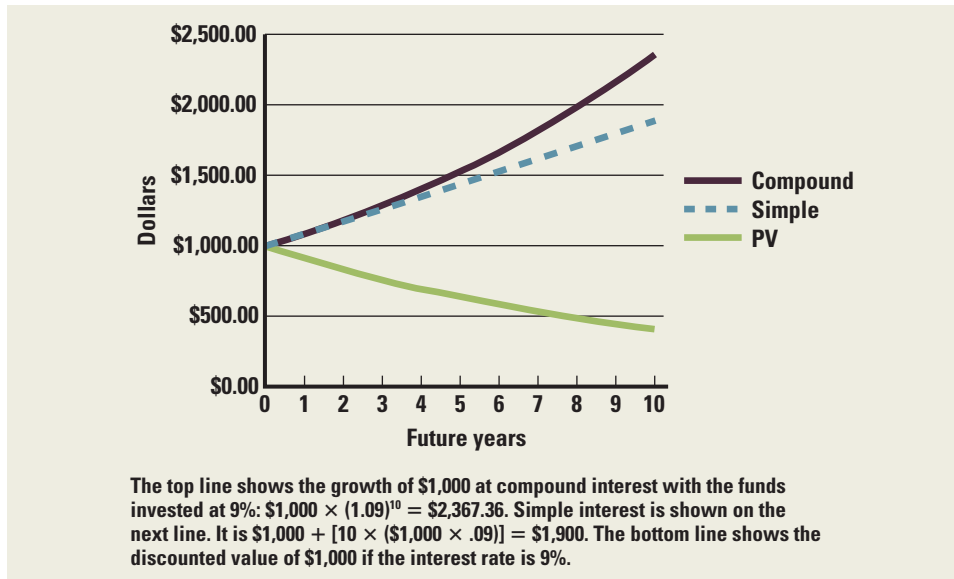In the preceding equation, PV stands for present value, the amount of money we must lend today in order to receive $1 in two years' time.

Solving for PV in this equation, we have

$$PV = \frac{\$1}{1.1881} = \$.84$$

This process of calculating the present value of a future cash flow is called **discounting**. It is the opposite of compounding. The difference between compounding and discounting is illustrated in Figure 4.8.

**FIGURE 4.8**
Compounding and Discounting

The top line shows the growth of $1,000 at compound interest with the funds invested at 9%: $1,000 × (1.09)^{10} = $2,367.36. Simple interest is shown on the next line. It is $1,000 + [10 × ($1,000 × .09)] = $1,900. The bottom line shows the discounted value of $1,000 if the interest rate is 9%.

To be certain that $.84 is in fact the present value of $1 to be received in two years, we must check whether or not, if we loaned out $.84 and rolled over the loan for two years, we would get exactly $1 back. If this were the case, the capital markets would be saying that $1 received in two years' time is equivalent to having $.84 today. Checking the exact numbers, we get

$$\$.84168 \times 1.09 \times 1.09 = \$1$$

In other words, when we have capital markets with a sure interest rate of 9 percent, we are indifferent between receiving $.84 today or $1 in two years. We have no reason to treat these two choices differently from each other, because if we had $.84 today and loaned it out for two years, it would return $1 to us at the end of that time. The value $[1/(1.09)^2]$ is called the **present value factor**. It is the factor used to calculate the present value of a future cash flow.

In the multiperiod case, the formula for PV can be written as

**Present Value of Investment:**

$$PV = \frac{C_T}{(1 + r)^T} \qquad [4.4]$$

where $C_T$ is cash flow at date $T$ and $r$ is the appropriate discount rate.

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**EXAMPLE 4.7**

**Multiperiod Discounting**

Bernard Dumas will receive $10,000 three years from now. Bernard can earn 8 percent on his investments, and so the appropriate discount rate is 8 percent. What is the present value of his future cash flow?
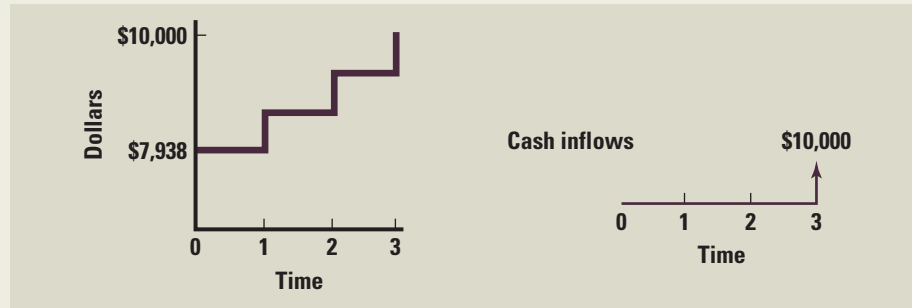
$$PV = \$10,000 \times \left(\frac{1}{1.08}\right)^3$$
$$= \$10,000 \times .7938$$
$$= \$7,938$$

(*continued*)

Figure 4.9 illustrates the application of the present value factor to Bernard's investment.

**FIGURE 4.9**
Discounting Bernard Dumas' Opportunity



When his investments grow at an 8 percent rate of interest, Bernard Dumas is equally inclined toward receiving $7,938 now and receiving $10,000 in three years' time. After all, he could convert the $7,938 he receives today into $10,000 in three years by lending it at an interest rate of 8 percent.

Bernard Dumas could have reached his present value calculation in one of three ways. The computation could have been done by hand, by calculator, or with the help of Table A.1, which appears in the back of the text. This table presents *present value of $1 to be received after T periods*. The table is used by locating the appropriate interest rate on the horizontal and the appropriate number of periods on the vertical. For example, Bernard Dumas would look at the following portion of Table A.1:

| | INTEREST RATE | | |
| PERIOD | 7% | 8% | 9% |
| --- | --- | --- | --- |
| 1 | .9346 | .9259 | .9174 |
| 2 | .8734 | .8573 | .8417 |
| 3 | .8163 | .7938 | .7722 |
| 4 | .7629 | .7350 | .7084 |

The appropriate present value factor is .7938.

In the preceding example, we gave both the interest rate and the future cash flow. Alternatively, the interest rate could have been unknown.

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**EXAMPLE 4.8**

**Finding the Rate**

A customer of the Chaffkin Corp. wants to buy a tugboat today. Rather than paying immediately, he will pay $50,000 in three years. It will cost the Chaffkin Corp. $38,610 to build the tugboat immediately. The relevant cash flows to Chaffkin Corp. are displayed in Figure 4.10. By charging what interest rate would the Chaffkin Corp. neither gain nor lose on the sale?

(*continued*)

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**FIGURE 4.10**
Cash Flows for Tugboat
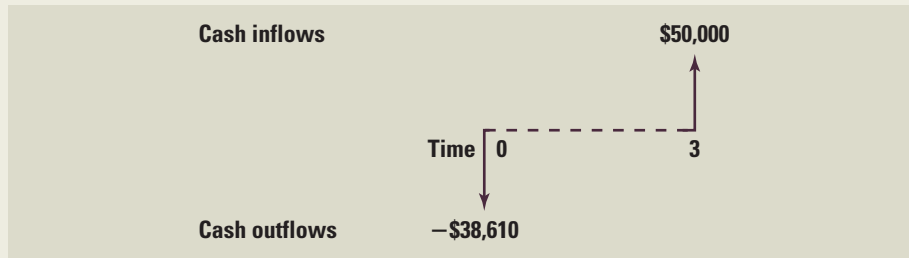


The ratio of construction cost to sale price is

$$\frac{\$38{,}610}{\$50{,}000} = 0.7722$$

We must determine the interest rate that allows \$1 to be received in three years to have a present value of \$.7722. Table A.1 tells us that 9 percent is that interest rate.

Frequently, an investor or a business will receive more than one cash flow. The present value of the set of cash flows is simply the sum of the present values of the individual cash flows. This is illustrated in the following examples.

**EXAMPLE 4.9**

**Cash Flow Valuation**

Dennis Draper has won the Kentucky state lottery and will receive the following set of cash flows over the next two years:

| YEAR | CASH FLOW |
|------|-----------|
| 1 | \$2,000 |
| 2 | \$5,000 |

Mr. Draper can currently earn 6 percent in his money market account, and so, the appropriate discount rate is 6 percent. The present value of the cash flows is

| YEAR | CASH FLOW × PRESENT VALUE FACTOR = PRESENT VALUE |
|------|--------------------------------------------------|
| 1 | $\$2{,}000 \times \frac{1}{1.06}$ = \$2,000 × .943 = \$1,887 |
| 2 | $\$5{,}000 \times \left(\frac{1}{1.06}\right)^{2}$ = \$5,000 × .890 = \$4,450 |
| | Total    \$6,337 |

In other words, Mr. Draper is equally inclined toward receiving \$6,337 today and receiving \$2,000 and \$5,000 over the next two years.

EXAMPLE 4.10

## NPV

Finance.com has an opportunity to invest in a new high-speed computer that costs $50,000. The computer will generate cash flows (from cost savings) of $25,000 one year from now, $20,000 two years from now, and $15,000 three years from now. The computer will be worthless after three years, and no additional cash flows will occur. Finance.com has determined that the appropriate discount rate is 7 percent for this investment. Should Finance.com make this investment in a new high-speed computer? What is the present value of the investment?

The cash flows and present value factors of the proposed computer are as follows.

|  | CASH FLOWS | PRESENT VALUE FACTOR | | |
|---|---|---|---|---|
| Year 0 | −$50,000 | $1$ | = | 1 |
| 1 | $25,000 | $\dfrac{1}{1.07}$ | = | .9346 |
| 2 | $20,000 | $\left(\dfrac{1}{1.07}\right)^2$ | = | .8734 |
| 3 | $15,000 | $\left(\dfrac{1}{1.07}\right)^3$ | = | .8163 |

The present values of the cash flows are:

Cash flows × Present value factor = Present value

| Year 0 | −$50,000 × 1 | = | −$50,000 |
|---|---|---|---|
| 1 | $25,000 × .9346 | = | $23,364.5 |
| 2 | $20,000 × .8734 | = | $17,468.8 |
| 3 | $15,000 × .8163 | = | $12,244.5 |
| | | Total | $ 3,077.8 |

Finance.com should invest in a new high-speed computer because the present value of its future cash flows is greater than its cost. The NPV is $3,077.8.

### The Algebraic Formula

To derive an algebraic formula for the net present value of a cash flow, recall that the PV of receiving a cash flow one year from now is

$$PV = C_1/(1 + r)$$

and the PV of receiving a cash flow two years from now is

$$PV = C_2/(1 + r)^2$$

We can write the NPV of a $T$-period project as

$$NPV = -C_0 + \frac{C_1}{1 + r} + \frac{C_2}{(1 + r)^2} + \cdots + \frac{C_T}{(1 + r)^T} = -C_0 + \sum_{i=1}^{T}\frac{C_i}{(1 + r)^i} \qquad [4.5]$$

The initial flow, $-C_0$, is assumed to be negative because it represents an investment. The $\sum$ is shorthand for the sum of the series.

We will close out this section by answering the question we posed at the beginning of the chapter concerning baseball player Matt Holliday's contract. Recall that the contract called for a salary of $17 million in each year over the next seven years, with $2 million in deferred salary. We will also assume that the option for 2017 is not picked up so he only receives $1 million in that year. The deferred salary payments from 2020 to 2029 could

actually be either $2 million or $3.2 million, depending on certain factors. In this case, we will assume that the deferred payments are $3.2 million per year. If 12 percent is the appropriate discount rate, what kind of deal did the Cardinal's outfielder catch?

To answer, we can calculate the present value by discounting each year's salary back to the present as follows (notice we assumed the future salaries will be paid at the end of the year):

| Year 1: | $15,000,000 \times 1/1.12^1$ | = | $13,392,857.14 |
|---------|------------------------------|---|----------------|
| Year 2: | $15,000,000 \times 1/1.12^2$ | = | $11,957,908.16 |
| Year 3: | $15,000,000 \times 1/1.12^3$ | = | $10,676,703.72 |
| Year 4: | $15,000,000 \times 1/1.12^4$ | = | $ 9,532,771.18 |
| | . | . | . |
| | . | . | . |
| | . | . | . |
| Year 20: | $ 3,200,000 \times 1/1.12^{20}$ | = | $   331,733.65 |

If you fill in the missing rows and then add (do it for practice), you will see that Matt's contract had a present value of about $74.68 million, which is only about 60 percent of the $120 million value reported, but still pretty good.

As you have probably noticed, doing extensive present value calculations can get to be pretty tedious, so a nearby *Spreadsheet Techniques* box shows how we recommend doing them. As an application, we take a look at lottery payouts in a *The Real World* box on page 100.

## 4.3 COMPOUNDING PERIODS

So far we have assumed that compounding and discounting occur yearly. Sometimes compounding may occur more frequently than just once a year. For example, imagine that a bank pays a 10-percent interest rate "compounded semiannually." This means that a $1,000 deposit in the bank would be worth $1,000 \times 1.05 = $1,050 after six months, and $1,050 \times 1.05 = $1,102.50 at the end of the year.

The end-of-the-year wealth can be written as

$$\$1,000 \left(1 + \frac{.10}{2}\right)^2 = \$1,000 \times (1.05)^2 = \$1,102.50$$

Of course, a $1,000 deposit would be worth $1,100 ($1,000 \times 1.10$) with yearly compounding. Note that the future value at the end of one year is greater with semiannual compounding than with yearly compounding. With yearly compounding, the original $1,000 remains the investment base for the full year. The original $1,000 is the investment base only for the first six months with semiannual compounding. The base over the second six months is $1,050. Hence, one gets *interest on interest* with semiannual compounding.

Because $1,000 \times 1.1025 = $1,102.50, 10 percent compounded semiannually is the same as 10.25 percent compounded annually. In other words, a rational investor could not care less whether she is quoted a rate of 10 percent compounded semiannually, or a rate of 10.25 percent compounded annually.

Quarterly compounding at 10 percent yields wealth at the end of one year of

$$\$1,000 \left(1 + \frac{.10}{4}\right)^4 = \$1,103.81$$

More generally, compounding an investment $m$ times a year provides end-of-year wealth of

$$C_0 \left(1 + \frac{r}{m}\right)^m \tag{4.6}$$

where $C_0$ is one's initial investment and $r$ is the **stated annual interest rate**. The stated annual interest rate is the annual interest rate without consideration of compounding.

We can set up a basic spreadsheet to calculate the present values of the individual cash flows as follows. Notice that we have simply calculated the present values one at a time and added them up:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | Using a spreadsheet to value multiple future cash flows | | | |
| 3 | | | | | |
| 4 | What is the present value of $200 in one year, $400 the next year, $600 the next year, and | | | | |
| 5 | $800 the last year if the discount rate is 12 percent? | | | | |
| 6 | | | | | |
| 7 | Rate: | 0.12 | | | |
| 8 | | | | | |
| 9 | Year | Cash flows | Present values | Formula used | |
| 10 | 1 | $200 | $178.57 | =PV($B$7,A10,0,−B10) | |
| 11 | 2 | $400 | $318.88 | =PV($B$7,A11,0,−B11) | |
| 12 | 3 | $600 | $427.07 | =PV($B$7,A12,0,−B12) | |
| 13 | 4 | $800 | $508.41 | =PV($B$7,A13,0,−B13) | |
| 14 | | | | | |
| 15 | | Total PV: | **$1,432.93** | =SUM(C10:C13) | |
| 16 | | | | | |
| 17 | Notice the negative signs inserted in the PV formulas. These just make the present values have | | | | |
| 18 | positive signs. Also, the discount rate in cell B7 is entered as $B$7 (an "absolute" reference) | | | | |
| 19 | because it is used over and over. We could have just entered ".12" instead, but our approach is more | | | | |
| 20 | flexible. | | | | |
| 21 | | | | | |
| 22 | | | | | |

Banks and other financial institutions may use other names for the stated annual interest rate. **Annual percentage rate (APR)** is perhaps the most common synonym.[3]

### EARs

What is the end-of-year wealth if Jane Christine receives a stated annual interest rate of 24 percent compounded monthly on a $1 investment?

Using (4.6), her wealth is

$$\$1\left(1 + \frac{.24}{12}\right)^{12} = \$1 \times (1.02)^{12}$$

$$= \$1.2682$$

The annual rate of return is 26.82 percent. This annual rate of return is either called the **effective annual rate (EAR)** or the **effective annual yield (EAY)**. Due to compounding, the effective annual

---

[3]By law, lenders are required to report the APR on all loans. In this text, we compute the APR as the interest rate per period multiplied by the number of periods in a year. According to federal law, the APR is a measure of the cost of consumer credit expressed as a yearly rate and it includes interest and certain noninterest charges and fees. In practice, the APR can be much higher than the interest rate on the loan if the lender charges substantial fees that must be included in the federally mandated APR calculation.

interest rate is greater than the stated annual interest rate of 24 percent. Algebraically, we can rewrite the effective annual interest rate as

**Effective Annual Rate:**

$$\left(1 + \frac{r}{m}\right)^m - 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[4.7]}$$

Students are often bothered by the subtraction of 1 in (4.7). Note that end-of-year wealth is composed of both the interest earned over the year and the original principal. We remove the original principal by subtracting 1 in (4.7).

### Compounding Frequencies

If the stated annual rate of interest, 8 percent, is compounded quarterly, what is the effective annual rate?

Using (4.7), we have

$$\left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{.08}{4}\right)^4 - 1 = .0824 = 8.24\%$$

Referring back to our earlier example where $C_0 = \$1,000$ and $r = 10\%$, we can generate the following table:

| $C_0$ | COMPOUNDING FREQUENCY ($m$) | $C_1$ | EFFECTIVE ANNUAL RATE = $\left(1 + \frac{r}{m}\right)^m - 1$ |
|---|---|---|---|
| $1,000 | Yearly ($m = 1$) | $1,100.00 | .10 |
| 1,000 | Semiannually ($m = 2$) | 1,102.50 | .1025 |
| 1,000 | Quarterly ($m = 4$) | 1,103.81 | .10381 |
| 1,000 | Daily ($m = 365$) | 1,105.16 | .10516 |

## Distinction between Stated Annual Interest Rate and Effective Annual Rate

The distinction between the stated annual interest rate (SAIR), or APR, and the effective annual rate (EAR) is frequently quite troubling to students. One can reduce the confusion by noting that the SAIR becomes meaningful only if the compounding interval is given. For example, for an SAIR of 10 percent, the future value at the end of one year with semiannual compounding is $[1 + (.10/2)]^2 = 1.1025$. The future value with quarterly compounding is $[1 + (.10/4)]^4 = 1.1038$. If the SAIR is 10 percent but no compounding interval is given, one cannot calculate future value. In other words, one does not know whether to compound semiannually, quarterly, or over some other interval.

By contrast, the EAR is meaningful *without* a compounding interval. For example, an EAR of 10.25 percent means that a $1 investment will be worth $1.1025 in one year. One can think of this as an SAIR of 10 percent with semiannual compounding or an SAIR of 10.25 percent with annual compounding, or some other possibility.

There can be a big difference between an SAIR and an EAR when interest rates are large. For example, consider "payday loans." Payday loans are short-term term loans made to consumers, often for less than two weeks, and are offered by companies such as Ameri-Cash Advance and National Payday. The loans work like this: you write a check today that is postdated. When the check date arrives, you go to the store and pay the cash for the check, or the company cashes the check. For example, AmeriCash Advance allows you to write a postdated check for $120 for 15 days later. In this case, they would give you $100

## JACKPOT!

If you or someone you know is a regular lottery player, you probably already understand that you are 20 times more likely to get struck by lightning than you are to win a big lottery jackpot. What are your odds of winning? Below you will find a table with your chances of winning the Mega Millions Lottery compared to other events.

| | |
|---|---:|
| Odds of winning a Mega Millions jackpot | 1:135,145,920* |
| Odds of being killed by a venomous spider | 1:57,018,763 |
| Odds of being killed by a dog bite | 1:11,403,753 |
| Odds of being killed by lightning | 1:6,479,405 |
| Odds of being killed by drowning | 1:690,300 |
| Odds of being killed falling from a bed or other furniture | 1:388,411 |
| Odds of being killed in a car crash | 1:6,029 |
| *Source: Virginia Lottery Web site. All other odds from the National Safety Council. | |

Sweepstakes may have different odds than lotteries, but these odds may not be much better. Probably the largest advertised potential grand prize ever was Pepsi's "Play for a Billion," which, you guessed it, had a $1 billion (*billion!*) prize. Not bad for a day's work, but you still have to read the fine print. It turns out that the winner would be paid $5 million per year for the next 20 years, $10 million per year for years 21 through 39, and a lump sum $710 million in 40 years. From what you have learned, you know the value of the sweepstakes wasn't even close to $1 billion. In fact, at an interest rate of 10 percent, the present value is about $70.7 million.

In January 2010, a 59-year-old man and his 57-year-old wife in New York won the $162 million Mega Millions jackpot. They were given the option of receiving the jackpot as $6.231 million immediately and $6.231 million per year for the next 25 years, or $102 million immediately. So, what discount rate does this imply? After some computational effort, we find the interest rate is about 4.15 percent. Unfortunately for the winners, nearly $1 million was placed in an escrow account over a dispute about the mismanagement of funds at a homeless shelter the couple had previously operated.

Some lotteries make your decision a little tougher. The Ontario Lottery will pay you either $2,000 a week for the rest of your life or $1.3 million now. (That's in Canadian dollars or "loonies," by the way.) Of course, there is the chance you might die in the near future, so the lottery guarantees that your heirs will collect the $2,000 weekly payments until the twentieth anniversary of the first payment, or until you would have turned 91, whichever comes first. This payout scheme complicates your decision quite a bit. If you live for only the 20-year minimum, the break-even interest rate between the two options is about 5.13 percent per year, compounded weekly. If you expect to live longer than the 20-year minimum, you might be better off accepting $2,000 per week for life. Of course, if you manage to invest the $1.3 million lump sum at a rate of return of about 8 percent per year (compounded weekly), you can have your cake and eat it too because the investment will return $2,000 at the end of each week forever! Taxes complicate the decision in this case because the lottery payments are all on an aftertax basis. Thus, the rates of return in this example would have to be aftertax as well.

today. So what is the APR and EAR of this arrangement? First we need to find the interest rate, which we can find by the FV equation as:

$$FV = PV (1 + r)^t$$
$$\$120 = \$100 \times (1 + r)^1$$
$$1.2 = (1 + r)$$
$$r = .20 \text{ or } 20\%$$

That doesn't seem too bad until you remember this is the interest rate for *15 days!* The APR of the loan is:

**APR = .20 × 365/15**
**APR = 4.8667 or 486.67%**

And the EAR for this loan is:

**EAR = (1 + Quoted rate/*m*)^*m* − 1**
**EAR = (1 + .20)^365/15 − 1**
**EAR = 83.4780 or 8,347.80%**

Now that's an interest rate! Just to see what a difference a day makes, let's look at another loan by the same company. AmeriCash Advance also offers a 14-day (instead of 15-day) option. The other terms are the same. Check for yourself that the APR of this arrangement is 521.43 percent and the EAR is 11,497.60 percent—definitely not a loan we recommend you take out!

## Compounding over Many Years

Formula 4.6 applies for an investment over one year. For an investment over one or more (*T*) years, the formula becomes

**Future Value with Compounding:**

$$FV = C_0\left(1 + \frac{r}{m}\right)^{mT}$$ [4.8]

**EXAMPLE 4.13**

**Multiyear Compounding**

Harry DeAngelo is investing $5,000 at a stated annual interest rate of 12 percent per year, compounded quarterly, for five years. What is his wealth at the end of five years?

Using formula (4.8), his wealth is

$$\$5,000 \times \left(1 + \frac{.12}{4}\right)^{4\times5} = \$5,000 \times (1.03)^{20} = \$5,000 \times 1.8061 = \$9,030.50$$

## Continuous Compounding

The previous discussion shows that one can compound much more frequently than once a year. One could compound semiannually, quarterly, monthly, daily, hourly, each minute, or even more often. The limiting case would be to compound every infinitesimal instant, which is commonly called **continuous compounding**. Surprisingly, banks and other financial institutions sometimes quote continuously compounded rates, which is why we study them.

Though the idea of compounding this rapidly may boggle the mind, a simple formula is involved. With continuous compounding, the value at the end of *T* years is expressed as

$$C_0 \times e^{rT}$$ [4.9]

where $C_0$ is the initial investment, *r* is the stated annual interest rate, and *T* is the number of years over which the investment runs. The number *e* is a constant and is approximately equal to 2.718. It is not an unknown like $C_0$, *r*, and *T*.

EXAMPLE 4.14

## Continuous Compounding

Linda DeFond invested $1,000 at a continuously compounded rate of 10 percent for one year. What is the value of her wealth at the end of one year?

From formula (4.9) we have

$1,000 \times e^{.10} = \$1,000 \times 1.1052 = \$1,105.20$

This number can easily be read from our Table A.5. One merely sets $r$, the value on the horizontal dimension, to 10 percent and $T$, the value on the vertical dimension, to 1. For this problem, the relevant portion of the table is

| PERIOD ($T$) | CONTINUOUSLY COMPOUNDED RATE ($r$) | | |
|---|---|---|---|
| | 9% | 10% | 11% |
| 1 | 1.0942 | 1.1052 | 1.1163 |
| 2 | 1.1972 | 1.2214 | 1.2461 |
| 3 | 1.3100 | 1.3499 | 1.3910 |

Note that a continuously compounded rate of 10 percent is equivalent to an annually compounded rate of 10.52 percent. In other words, Linda DeFond would not care whether her bank quoted a continuously compounded rate of 10 percent or a 10.52-percent rate, compounded annually.

EXAMPLE 4.15

## Continuous Compounding, Continued

Linda DeFond's brother, Mark, invested $1,000 at a continuously compounded rate of 10 percent for two years.

The appropriate formula here is

$1,000 \times e^{.10 \times 2} = \$1,000 \times e^{.20} = \$1,221.40$

Using the portion of the table of continuously compounded rates reproduced above, we find the value to be 1.2214.

Figure 4.11 illustrates the relationship among annual, semiannual, and continuous compounding. Semiannual compounding gives rise to both a smoother curve and a higher ending value than does annual compounding. Continuous compounding has both the smoothest curve and the highest ending value of all.

**FIGURE 4.11**

Annual, Semiannual, and Continuous Compounding



Annual compounding    Semiannual compounding    Continuous compounding

**EXAMPLE 4.16**
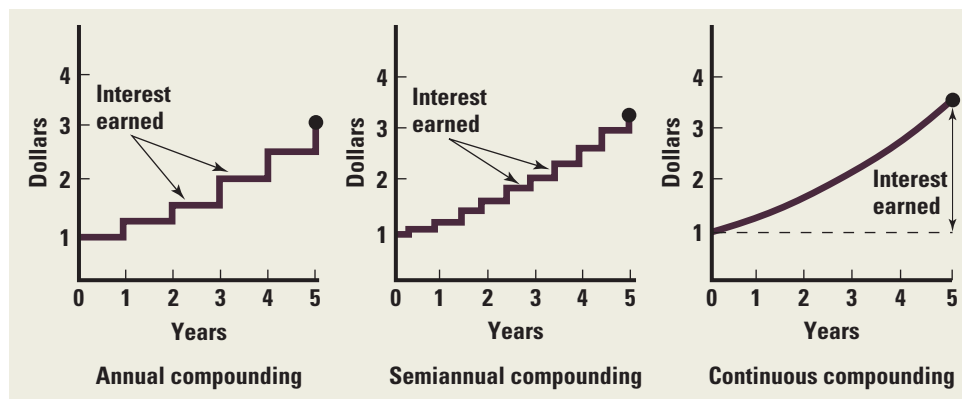
The Michigan state lottery is going to pay you $1,000 at the end of four years. If the annual continuously compounded rate of interest is 8 percent, what is the present value of this payment?

$$\$1,000 \times \frac{1}{e^{.08 \times 4}} = \$1,000 \times \frac{1}{1.3771} = \$726.16$$

## 4.4 SIMPLIFICATIONS

The first part of this chapter has examined the concepts of future value and present value. Although these concepts allow one to answer a host of problems concerning the time value of money, the human effort involved can frequently be excessive. For example, consider a bank calculating the present value on a 20-year monthly mortgage. Because this mortgage has 240 (20 × 12) payments, a lot of time is needed to perform a conceptually simple task.

Because many basic finance problems are potentially so time-consuming, we search out simplifications in this section. We provide simplifying formulas for four classes of cash flow streams:

- Perpetuity
- Growing perpetuity
- Annuity
- Growing annuity

### Perpetuity

A **perpetuity** is a constant stream of cash flows without end. If you are thinking that perpetuities have no relevance to reality, it will surprise you that there is a well-known case of an unending cash flow stream: the British bonds called *consols*. An investor purchasing a consol is entitled to receive yearly interest from the British government forever.

How can the price of a consol be determined? Consider a consol that pays a coupon of $C$ dollars each year and will do so forever. Simply applying the PV formula gives us

$$\text{PV} = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \frac{C}{(1 + r)^3} + \cdots$$

where the dots at the end of the formula stand for the infinite string of terms that continues the formula. Series like the preceding one are called *geometric series*. It is well known that even though they have an infinite number of terms, the whole series has a finite sum because each term is only a fraction of the preceding term. Before turning to our calculus books, though, it is worth going back to our original principles to see if a bit of financial intuition can help us find the PV.

The present value of the consol is the present value of all of its future coupons. In other words, it is an amount of money that, if an investor had it today, would enable him to achieve the same pattern of expenditures that the consol and its coupons would. Suppose that an investor wanted to spend exactly $C$ dollars each year. If he had the consol, he could do this. How much money must he have today to spend the same amount? Clearly he would need exactly enough so that the interest on the money would be $C$ dollars per year. If he had any more, he could spend more than $C$ dollars each year. If he had any less, he would eventually run out of money spending $C$ dollars per year.

The amount that will give the investor $C$ dollars each year, and therefore the present value of the consol, is simply

$$\text{PV} = \frac{C}{r} \qquad\qquad\qquad [4.10]$$

To confirm that this is the right answer, notice that if we lend the amount $C/r$, the interest it earns each year will be

$$\text{Interest} = \frac{C}{r} \times r = C$$

which is exactly the consol payment. To sum up, we have shown that for a consol

**Formula for Present Value of Perpetuity:**

$$\text{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \qquad\qquad [4.11]$$

$$= \frac{C}{r}$$

It is comforting to know how easily we can use a bit of financial intuition to solve this mathematical problem.

---

**EXAMPLE 4.17**

**Perpetuities**

Consider a perpetuity paying $100 a year. If the relevant interest rate is 8 percent, what is the value of the consol?

Using formula (4.10), we have

$$\text{PV} = \frac{\$100}{.08} = \$1,250$$

Now suppose that interest rates fall to 6 percent. Using (4.10), the value of the perpetuity is

$$\text{PV} = \frac{\$100}{.06} = \$1,666.67$$

Note that the value of the perpetuity rises with a drop in the interest rate. Conversely, the value of the perpetuity falls with a rise in the interest rate.

---

## Growing Perpetuity

Imagine an apartment building where cash flows to the landlord after expenses will be $100,000 next year. These cash flows are expected to rise at 5 percent per year. If one assumes that this rise will continue indefinitely, the cash flow stream is termed a **growing perpetuity**. The relevant interest rate is 11 percent. Therefore, the appropriate discount rate is 11 percent and the present value of the cash flows can be represented as

$$\text{PV} = \frac{\$100,000}{1.11} + \frac{\$100,000(1.05)}{(1.11)^2} + \frac{\$100,000(1.05)^2}{(1.11)^3} + \cdots$$

$$+ \frac{\$100,000(1.05)^{N-1}}{(1.11)^N} + \cdots$$

Algebraically, we can write the formula as

$$\text{PV} = \frac{C}{1+r} + \frac{C \times (1+g)}{(1+r)^2} + \frac{C \times (1+g)^2}{(1+r)^3} + \cdots + \frac{C \times (1+g)^{N-1}}{(1+r)^N} + \cdots$$

where $C$ is the cash flow to be received one period hence, $g$ is the rate of growth per period, expressed as a percentage, and $r$ is the appropriate discount rate.

Fortunately, this formula reduces to the following simplification:

**Formula for Present Value of Growing Perpetuity:**

$$\text{PV} = \frac{C}{r-g} \qquad\qquad\qquad [4.12]$$

From Formula 4.12, the present value of the cash flows from the apartment building is

$$\frac{\$100,000}{.11 - .05} = \$1,666,667$$

There are three important points concerning the growing perpetuity formula:

1. *The Numerator*. The numerator in Formula 4.12 is the cash flow one period hence, not at date 0. Consider the following example:

**Paying Dividends**

Rothstein Corporation is *just about* to pay a dividend of $3.00 per share. Investors anticipate that the annual dividend will rise by 6 percent a year forever. The applicable discount rate is 11 percent. What is the price of the stock today?

    The numerator in Formula 4.12 is the cash flow to be received next period. Since the growth rate is 6 percent, the dividend next year is $3.18 ($3.00 × 1.06). The price of the stock today is

| $66.60 | = | $3.00 | + | $\dfrac{\$3.18}{.11 - .06}$ |
|---|---|---|---|---|
| | | Imminent dividend | | Present value of all dividends beginning a year from now |

The price of $66.60 includes both the dividend to be received immediately and the present value of all dividends beginning a year from now. Formula 4.12 only makes it possible to calculate the present value of all dividends beginning a year from now. Be sure you understand this example; test questions on this subject always seem to trip up a few of our students.
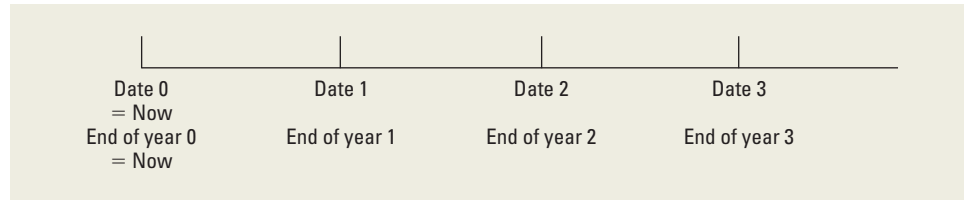
2. *The Discount Rate and the Growth Rate*. The discount rate $r$ must be greater than the growth rate $g$ for the growing perpetuity formula to work. Consider the case in which the growth rate approaches the discount rate in magnitude. Then the denominator in the growing perpetuity formula gets infinitesimally small and the present value grows infinitely large. The present value is in fact undefined when $r$ is less than $g$.

3. *The Timing Assumption*. Cash generally flows into and out of real-world firms both randomly and nearly continuously. However, Formula 4.12 assumes that cash flows are received and disbursed at regular and discrete points in time. In the example of the apartment, we assumed that the net cash flows of $100,000 only occurred once a year. In reality, rent checks are commonly received every month. Payments for maintenance and other expenses may occur anytime within the year.

    The growing perpetuity formula (4.12) can be applied only by assuming a regular and discrete pattern of cash flow. Although this assumption is sensible because the formula saves so much time, the user should never forget that it is an *assumption*. This point will be mentioned again in the chapters ahead.

    A few words should be said about terminology. Authors of financial textbooks generally use one of two conventions to refer to time. A minority of financial writers treat cash flows as being received on exact *dates*, for example date 0, date 1, and so forth. Under this convention, date 0 represents the present time. However, because a year is an interval, not a specific moment in time, the great majority of authors refer to cash flows that occur at the end of a year (or alternatively, the end of a *period*). Under this *end-of-the-year* convention,

the end of year 0 is the present, the end of year 1 occurs one period hence, and so on. (The beginning of year 0 has already passed and is not generally referred to.)[4]

The interchangeability of the two conventions can be seen from the following chart:

| | | | |
|---|---|---|---|
| Date 0 = Now | Date 1 | Date 2 | Date 3 |
| End of year 0 = Now | End of year 1 | End of year 2 | End of year 3 |

We strongly believe that the *dates convention* reduces ambiguity. However, we use both conventions because you are likely to see the *end-of-year convention* in later courses. In fact, both conventions may appear in the same example for the sake of practice.

## Annuity

An **annuity** is a level stream of regular payments that lasts for a fixed number of periods. Not surprisingly, annuities are among the most common kinds of financial instruments. The pensions that people receive when they retire are often in the form of an annuity. Leases and mortgages are also often annuities.

To figure out the present value of an annuity we need to evaluate the following equation:

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^T}$$

The present value of only receiving the coupons for $T$ periods must be less than the present value of a consol, but how much less? To answer this we have to look at consols a bit more closely.

Consider the following time chart:

| Date (or end of year) | Now<br>0 | 1 | 2 | 3 | $T$ | | $(T+1)$ | $(T+2)$ |
|---|---|---|---|---|---|---|---|---|
| Consol 1 | | $C$ | $C$ | $C\ldots$ | $C$ | | $C$ | $C\ldots$ |
| Consol 2 | | | | | | | $C$ | $C\ldots$ |
| Annuity | | $C$ | $C$ | $C\ldots$ | $C$ | | | |

Consol 1 is a normal consol with its first payment at date 1. The first payment of consol 2 occurs at date $T + 1$.

The present value of having a cash flow of $C$ at each of $T$ dates is equal to the present value of consol 1 minus the present value of consol 2. The present value of consol 1 is given by

$$\text{PV} = \frac{C}{r} \qquad\qquad [4.13]$$

Consol 2 is just a consol with its first payment at date $T + 1$. From the perpetuity formula, this consol will be worth $C/r$ at date $T$.[5] However, we do not want the value at date $T$. We

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[4]Sometimes financial writers merely speak of a cash flow in year *x*. Although this terminology is ambiguous, such writers generally mean the *end of year x*.

[5]Students frequently think that $C/r$ is the present value at date $T + 1$ because the consol's first payment is at date $T + 1$. However, the formula values the annuity as of one period prior to the first payment.

want the value now; in other words, the present value at date 0. We must discount $C/r$ back by $T$ periods. Therefore, the present value of consol 2 is

$$PV = \frac{C}{r}\left[\frac{1}{(1+r)^T}\right] \qquad [4.14]$$

The present value of having cash flows for $T$ years is the present value of a consol with its first payment at date 1 minus the present value of a consol with its first payment at date $T + 1$. Thus, the present value of an annuity is Formula 4.13 minus Formula 4.14. This can be written as

$$\frac{C}{r} - \frac{C}{r}\left[\frac{1}{(1+r)^T}\right]$$

This simplifies to

**Formula for Present Value of Annuity:**
$$PV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right] \qquad [4.15]$$

This can also be written as

$$PV = C\left[\frac{1 - \frac{1}{(1+r)^T}}{r}\right]$$

---

**EXAMPLE 4.19**

### Lottery Valuation

Mark Young has just won the state lottery, paying $50,000 a year for 20 years. He is to receive his first payment a year from now. The state advertises this as the Million Dollar Lottery because $1,000,000 = $50,000 × 20$. If the interest rate is 8 percent, what is the true value of the lottery?

Formula 4.15 yields

$$\begin{array}{c}\text{Present value of} \\ \text{Million Dollar Lottery} = \$50,000 \times \left[\frac{1 - \frac{1}{(1.08)^{20}}}{.08}\right]\end{array}$$

$$\begin{array}{ccc}\text{Periodic payment} & & \text{Annuity factor} \\ = \$50,000 & \times & 9.8181 \\ = \$490,905 & & \end{array}$$

Rather than being overjoyed at winning, Mr. Young sues the state for misrepresentation and fraud. His legal brief states that he was promised $1 million but received only $490,905.

---

The term we use to compute the present value of the stream of level payments, $C$, for $T$ years is called an **annuity factor**. The annuity factor in the current example is 9.8181. Because the annuity factor is used so often in PV calculations, we have included it in Table A.2 in the back of this book. The table gives the values of these factors for a range of interest rates, $r$, and maturity dates, $T$.

The annuity factor as expressed in the brackets of Formula 4.15 is a complex formula. For simplification, we may from time to time refer to the present value annuity factor as

$$PVIFA_{r,T}$$

That is, the above expression stands for the present value of $1 a year for $T$ years at an interest rate of $r$.

We can also provide a formula for the future value of an annuity:

$$FV = C\left[\frac{(1+r)^T}{r} - \frac{1}{r}\right] = C\left[\frac{(1+r)^T - 1}{r}\right] \qquad [4.16]$$

As with present value factors for annuities, we have compiled future value factors in Table A.3 in the back of this book. Of course, you can also use a spreadsheet as we illustrate below.

**EXAMPLE 4.20**

### Retirement Investing

Suppose you put $3,000 per year into a Roth IRA. The account pays 6 percent per year. How much will you have when you retire in 30 years?

This question asks for the future value of an annuity of $3,000 per year for 30 years at 6 percent, which we can calculate as follows:

$$FV = C\left[\frac{(1+r)^T - 1}{r}\right] = \$3,000 \times \left[\frac{1.06^{30} - 1}{.06}\right]$$

$$= \$3,000 \times 79.0582$$

$$= \$237,174.56$$

So, you'll have close to a quarter million dollars in the account.

Our experience is that annuity formulas are not hard, but tricky, for the beginning student. We present four tricks below.

**TRICK 1: A DELAYED ANNUITY**　　One of the tricks in working with annuities or perpetuities is getting the timing exactly right. This is particularly true when an annuity or perpetuity begins at a date many periods in the future. We have found that even the brightest beginning student can make errors here. Consider the following example.

**EXAMPLE 4.21**

**Delayed Annuities**

Danielle Caravello will receive a four-year annuity of $500 per year, beginning at date 6. If the interest rate is 10 percent, what is the present value of her annuity? This situation can be graphed as:
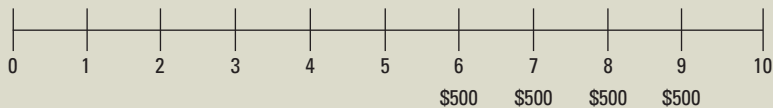


The analysis involves two steps:

1. Calculate the present value of the annuity using Formula 4.15. This is

**Present Value of Annuity at Date 5:**

$$\$500 \times \left[ \frac{1 - \frac{1}{(1.10)^4}}{.10} \right] = \$500 \times \text{PVIFA}_{10\%,4}$$

$$= \$500 \times 3.1699$$

$$= \$1,584.95$$

Note that $1,584.95 represents the present value at *date 5*.

Students frequently think that $1,584.95 is the present value at date 6, because the annuity begins at date 6. However, our formula values the annuity as of one period prior to the first payment. This can be seen in the most typical case where the first payment occurs at date 1. The formula values the annuity as of date 0 in that case.

2. Discount the present value of the annuity back to date 0. That is

**Present Value at Date 0:**

$$\frac{\$1,584.95}{(1.10)^5} = \$984.13$$

Again, it is worthwhile mentioning that, because the annuity formula brings Danielle's annuity back to date 5, the second calculation must discount over the remaining 5 periods. The two-step procedure is graphed in Figure 4.12.

**FIGURE 4.12**

Discounting Danielle Caravello's Annuity
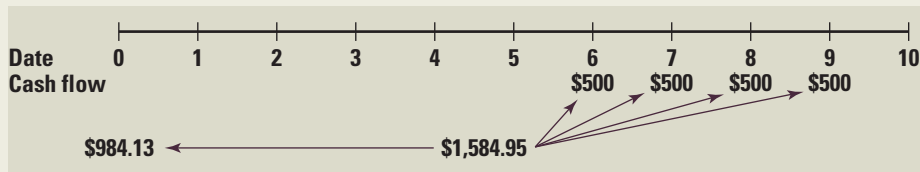


**Step one:** Discount the four payments back to date 5 by using the annuity formula.
**Step two:** Discount the present value at date 5 ($1,584.95) back to present value at date 0.

**TRICK 2: ANNUITY DUE** The annuity formula of Formula 4.15 assumes that the first annuity payment begins a full period hence. This type of annuity is sometimes called an *annuity in arrears* or an *ordinary annuity*. What happens if the annuity begins today, in other words, at date 0?

## Annuity Due

In a previous example, Mark Young received $50,000 a year for 20 years from the state lottery. In that example, he was to receive the first payment a year from the winning date. Let us now assume that the first payment occurs immediately. The total number of payments remains 20.

Under this new assumption, we have a 19-date annuity with the first payment occurring at date 1-plus an extra payment at date 0. The present value is

$$\begin{array}{ccc} \$50,000 & + & \$50,000 \times PVIFA_{8\%,19} \\ \text{Payment at date 0} & & \text{19-year annuity} \end{array}$$

$$= \$50,000 + (\$50,000 \times 9.6036)$$
$$= \$530,180$$

$530,180, the present value in this example, is greater than $490,905, the present value in the earlier lottery example. This is to be expected because the annuity of the current example begins earlier. An annuity with an immediate initial payment is called an *annuity in advance* or, more commonly, an *annuity due*. Always remember that Formula 4.15, as well as Table A.2, in this book refers to an *ordinary annuity*.

**TRICK 3: THE INFREQUENT ANNUITY**  The following example treats an annuity with payments occurring less frequently than once a year.

## Infrequent Annuities

Ms. Ann Chen receives an annuity of $450, payable once every two years. The annuity stretches out over 20 years. The first payment occurs at date 2, that is, two years from today. The annual interest rate is 6 percent.

The trick is to determine the interest rate over a two-year period. The interest rate over two years is

$$(1.06 \times 1.06) - 1 = 12.36\%$$

That is, $100 invested over two years will yield $112.36.

What we want is the present value of a $450 annuity over 10 periods, with an interest rate of 12.36 percent per period. This is

$$\$450 \times \left[ \frac{1 - \frac{1}{(1 + .1236)^{10}}}{.1236} \right] = \$450 \times PVIFA_{12.36\%,10} = \$2,505.57$$

**TRICK 4: EQUATING PRESENT VALUE OF TWO ANNUITIES**  The following example equates the present value of inflows with the present value of outflows.

## Working with Annuities

Harold and Helen Nash are saving for the college education of their newborn daughter, Susan. The Nashes estimate that college expenses will run $30,000 per year when their daughter reaches college in 18 years. The annual interest rate over the next few decades will be 14 percent. How much money must they deposit in the bank each year so that their daughter will be completely supported through four years of college?

To simplify the calculations, we assume that Susan is born today. Her parents will make the first of her four annual tuition payments on her 18th birthday. They will make equal bank deposits on each of her first 17 birthdays, but no deposit at date 0. This is illustrated as

| Date | 0 | 1 | 2 | ... | 17 | 18 | 19 | 20 | 21 |
|------|---|---|---|-----|-----|-----|-----|-----|-----|
| | Susan's birth | Parents' 1st deposit | Parents' 2nd deposit | ... | Parents' 17th and last deposit | Tuition payment 1 | Tuition payment 2 | Tuition payment 3 | Tuition payment 4 |
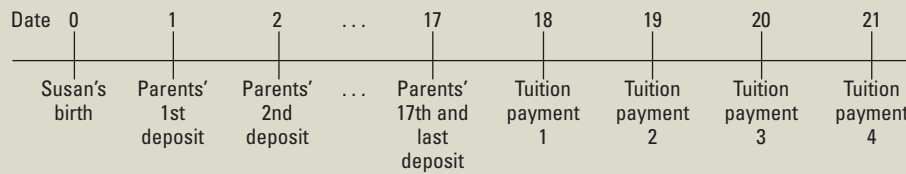
Mr. and Ms. Nash will be making deposits to the bank over the next 17 years. They will be withdrawing $30,000 per year over the following four years. We can be sure they will be able to withdraw fully $30,000 per year if the present value of the deposits is equal to the present value of the four $30,000 withdrawals.

This calculation requires three steps. The first two determine the present value of the withdrawals. The final step determines yearly deposits that will have a present value equal to that of the withdrawals.

1. We calculate the present value of the four years at college using the annuity formula.

$$\$30,000 \times \left[ \frac{1 - \frac{1}{(1.14)^4}}{.14} \right] = \$30,000 \times \text{PVIFA}_{14\%,4}$$
$$= \$30,000 \times 2.9137 = \$87,411$$

We assume that Susan enters college on her 18th birthday. Given our discussion in Trick 1, $87,411 represents the present value at date 17.

2. We calculate the present value of the college education at date 0 as

$$\frac{\$87,411}{(1.14)^{17}} = \$9,422.91$$

3. Assuming that Helen and Harold Nash make deposits to the bank at the end of each of the 17 years, we calculate the annual deposit that will yield a present value of all deposits of $9,422.91. This is calculated as

$$C \times \text{PVIFA}_{14\%,17} = \$9,422.91$$

Because $\text{PVIFA}_{14\%,17} = 6.3729$,

$$C = \frac{\$9,422.91}{6.3729} = \$1,478.59$$

Thus, deposits of $1,478.59 made at the end of each of the first 17 years and invested at 14 percent will provide enough money to make tuition payments of $30,000 over the following four years.

An alternative method would be to (1) calculate the present value of the tuition payments at Susan's 18th birthday and (2) calculate annual deposits such that the future value of the deposits at her 18th birthday equals the present value of the tuition payments at that date. Although this technique can also provide the right answer, we have found that it is more likely to lead to errors. Therefore, we only equate present values in our presentation.

## Growing Annuity

Cash flows in business are very likely to grow over time, due either to real growth or to inflation. The growing perpetuity, which assumes an infinite number of cash flows, provides

one formula to handle this growth. We now consider a **growing annuity**, which is a *finite* number of growing cash flows. Because perpetuities of any kind are rare, a formula for a growing annuity would be useful indeed. The formula is

**Formula for Present Value of Growing Annuity:**

$$\text{PV} = C\left[\frac{1}{r-g} - \frac{1}{r-g} \times \left(\frac{1+g}{1+r}\right)^T\right] = C\left[\frac{1 - \left(\frac{1+g}{1+r}\right)^T}{r-g}\right] \tag{4.17}$$

where, as before, $C$ is the payment to occur at the end of the first period, $r$ is the interest rate, $g$ is the rate of growth per period, expressed as a percentage, and $T$ is the number of periods for the annuity.

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## EXAMPLE 4.25

### Growing Annuities

Stuart Gabriel, a second-year MBA student, has just been offered a job at $80,000 a year. He anticipates his salary increasing by 9 percent a year until his retirement in 40 years. Given an interest rate of 20 percent, what is the present value of his lifetime salary?

We simplify by assuming he will be paid his $80,000 salary exactly one year from now, and that his salary will continue to be paid in annual installments. The appropriate discount rate is 20 percent. From (4.17), the calculation is

$$\begin{array}{l}\text{Present value}\\\quad\text{of Stuart's}\\\text{lifetime salary}\end{array} = \$80,000 \times \left[\frac{1 - \left(\frac{1.09}{1.20}\right)^{40}}{.20 - .09}\right] = \$711,731$$

Though the growing annuity is quite useful, it is more tedious than the other simplifying formulas. Whereas most sophisticated calculators have special programs for perpetuity, growing perpetuity, and annuity, there is no special program for growing annuity. Hence, one must calculate all the terms in Formula 4.17 directly.

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## EXAMPLE 4.26

### More Growing Annuities

In a previous example, Harold and Helen Nash planned to make 17 identical payments in order to fund the college education of their daughter, Susan. Alternatively, imagine that they planned to increase their payments at 4 percent per year. What would their first payment be?

The first two steps of the previous Nash family example showed that the present value of the college costs was $9,422.91. These two steps would be the same here. However, the third step must be altered. Now we must ask, How much should their first payment be so that, if payments increase by 4 percent per year, the present value of all payments will be $9,422.91?

We set the growing-annuity formula equal to $9,422.91 and solve for $C$.

$$C\left[\frac{1 - \left(\frac{1+g}{1+r}\right)^T}{r-g}\right] = C\left[\frac{1 - \left(\frac{1.04}{1.14}\right)^{17}}{.14 - .04}\right]$$
$$= \$9,422.91$$

Here, $C = \$1,192.78$. Thus, the deposit on their daughter's first birthday is $1,192.78, the deposit on the second birthday is $1,240.49 ($1.04 \times \$1,192.78$), and so on.

## 4.5 LOAN TYPES AND LOAN AMORTIZATION

Whenever a lender extends a loan, some provision will be made for repayment of the principal (the original loan amount). A loan might be repaid in equal installments, for example, or it might be repaid in a single lump sum. Because the way that the principal and interest are paid is up to the parties involved, there are actually an unlimited number of possibilities.

In this section, we describe a few forms of repayment that come up quite often, and more complicated forms can usually be built up from these. The three basic types of loans are pure discount loans, interest-only loans, and amortized loans. Working with these loans is a very straightforward application of the present value principles that we have already developed.

### Pure Discount Loans

The *pure discount loan* is the simplest form of loan. With such a loan, the borrower receives money today and repays a single lump sum at some time in the future. A one-year, 10 percent pure discount loan, for example, would require the borrower to repay $1.10 in one year for every dollar borrowed today.

Because a pure discount loan is so simple, we already know how to value one. Suppose a borrower was able to repay $25,000 in five years. If we, acting as the lender, wanted a 12 percent interest rate on the loan, how much would we be willing to lend? Put another way, what value would we assign today to that $25,000 to be repaid in five years? Based on our previous work we know the answer is just the present value of $25,000 at 12 percent for five years:

$$\text{Present value} = \$25,000/1.12^5$$
$$= \$25,000/1.7623$$
$$= \$14,186$$

Pure discount loans are common when the loan term is short, say a year or less. In recent years, they have become increasingly common for much longer periods.

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### Interest-Only Loans

A second type of loan repayment plan calls for the borrower to pay interest each period and to repay the entire principal (the original loan amount) at some point in the future. Loans with such a repayment plan are called *interest-only loans*. Notice that if there is just one period, a pure discount loan and an interest-only loan are the same thing.

For example, with a three-year, 10 percent, interest-only loan of $1,000, the borrower would pay $1,000 × .10 = $100 in interest at the end of the first and second years. At the

end of the third year, the borrower would return the $1,000 along with another $100 in interest for that year. Similarly, a 50-year interest-only loan would call for the borrower to pay interest every year for the next 50 years and then repay the principal. In the extreme, the borrower pays the interest every period forever and never repays any principal. As we discussed earlier in the chapter, the result is a perpetuity.

Most corporate bonds have the general form of an interest-only loan. Because we will be considering bonds in some detail in the next chapter, we will defer further discussion of them for now.

### Amortized Loans

With a pure discount or interest-only loan, the principal is repaid all at once. An alternative is an *amortized loan*, with which the lender may require the borrower to repay parts of the loan amount over time. The process of providing for a loan to be paid off by making regular principal reductions is called *amortizing* the loan.

A simple way of amortizing a loan is to have the borrower pay the interest each period plus some fixed amount. This approach is common with medium-term business loans. For example, suppose a business takes out a $5,000, five-year loan at 9 percent. The loan agreement calls for the borrower to pay the interest on the loan balance each year and to reduce the loan balance each year by $1,000. Because the loan amount declines by $1,000 each year, it is fully paid in five years.

In the case we are considering, notice that the total payment will decline each year. The reason is that the loan balance goes down, resulting in a lower interest charge each year, whereas the $1,000 principal reduction is constant. For example, the interest in the first year will be $5,000 × .09 = $450. The total payment will be $1,000 + 450 = $1,450. In the second year, the loan balance is $4,000, so the interest is $4,000 × .09 = $360, and the total payment is $1,360. We can calculate the total payment in each of the remaining years by preparing a simple *amortization schedule* as follows:

| YEAR | BEGINNING BALANCE | TOTAL PAYMENT | INTEREST PAID | PRINCIPAL PAID | ENDING BALANCE |
|---|---|---|---|---|---|
| 1 | $5,000 | $1,450 | $ 450 | $1,000 | $4,000 |
| 2 | 4,000 | 1,360 | 360 | 1,000 | 3,000 |
| 3 | 3,000 | 1,270 | 270 | 1,000 | 2,000 |
| 4 | 2,000 | 1,180 | 180 | 1,000 | 1,000 |
| 5 | 1,000 | 1,090 | 90 | 1,000 | 0 |
| Totals | | $6,350 | $1,350 | $5,000 | |

Notice that in each year, the interest paid is given by the beginning balance multiplied by the interest rate. Also notice that the beginning balance is given by the ending balance from the previous year.

Probably the most common way of amortizing a loan is to have the borrower make a single, fixed payment every period. Almost all consumer loans (such as car loans) and mortgages work this way. For example, suppose our five-year, 9 percent, $5,000 loan was amortized this way. How would the amortization schedule look?

We first need to determine the payment. From our discussion earlier in the chapter, we know that this loan's cash flows are in the form of an ordinary annuity. In this case, we can solve for the payment as follows:

$$\$5,000 = C \times \{[1 - (1/1.09^5)]/.09\}$$
$$= C \times [(1 - .6499)/.09]$$

This gives us:

$$C = \$5,000/3.8897$$
$$= \$1,285.46$$

The borrower will therefore make five equal payments of $1,285.46. Will this pay off the loan? We will check by filling in an amortization schedule.

In our previous example, we knew the principal reduction each year. We then calculated the interest owed to get the total payment. In this example, we know the total payment. We will thus calculate the interest and then subtract it from the total payment to calculate the principal portion in each payment.

In the first year, the interest is $450, as we calculated before. Because the total payment is $1,285.46, the principal paid in the first year must be:

**Principal paid = $1,285.46 − 450 = $835.46**

The ending loan balance is thus:

**Ending balance = $5,000 − 835.46 = $4,164.54**

The interest in the second year is $4,164.54 × .09 = $374.81, and the loan balance declines by $1,285.46 − 374.81 = $910.65. We can summarize all of the relevant calculations in the following schedule:

| YEAR | BEGINNING BALANCE | TOTAL PAYMENT | INTEREST PAID | PRINCIPAL PAID | ENDING BALANCE |
|------|-------------------|---------------|---------------|----------------|----------------|
| 1 | $5,000.00 | $1,285.46 | $ 450.00 | $ 835.46 | $4,164.54 |
| 2 | 4,164.54 | 1,285.46 | 374.81 | 910.65 | 3,253.88 |
| 3 | 3,253.88 | 1,285.46 | 292.85 | 992.61 | 2,261.27 |
| 4 | 2,261.27 | 1,285.46 | 203.51 | 1,081.95 | 1,179.32 |
| 5 | 1,179.32 | 1,285.46 | 106.14 | 1,179.32 | 0.00 |
| Totals | | $6,427.30 | $1,427.31 | $5,000.00 | |

Because the loan balance declines to zero, the five equal payments do pay off the loan. Notice that the interest paid declines each period. This isn't surprising because the loan balance is going down. Given that the total payment is fixed, the principal paid must be rising each period. To see how to calculate this loan in Excel, see the upcoming Spreadsheet Strategies box.

If you compare the two loan amortizations in this section, you will see that the total interest is greater for the equal total payment case: $1,427.31 versus $1,350. The reason for this is that the loan is repaid more slowly early on, so the interest is somewhat higher. This doesn't mean that one loan is better than the other; it simply means that one is effectively paid off faster than the other. For example, the principal reduction in the first year is $835.46 in the equal total payment case as compared to $1,000 in the first case.

**EXAMPLE 4.28**

**Partial Amortization, or "Bite the Bullet"**

A common arrangement in real estate lending might call for a 5-year loan with, say, a 15-year amortization. What this means is that the borrower makes a payment every month of a fixed amount based on a 15-year amortization. However, after 60 months, the borrower makes a single, much larger payment called a "balloon" or "bullet" to pay off the loan. Because the monthly payments don't fully pay off the loan, the loan is said to be partially amortized.

(*continued*)

Suppose we have a $100,000 commercial mortgage with a 12 percent APR and a 20-year (240-month) amortization. Further suppose the mortgage has a five-year balloon. What will the monthly payment be? How big will the balloon payment be?

The monthly payment can be calculated based on an ordinary annuity with a present value of $100,000. There are 240 payments, and the interest rate is 1 percent per month. The payment is:

$$\$100,000 = C \times [(1 - 1/1.01^{240})/.01]$$
$$= C \times 90.8194$$
$$C = \$1,101.09$$

Now, there is an easy way and a hard way to determine the balloon payment. The hard way is to actually amortize the loan for 60 months to see what the balance is at that time. The easy way is to recognize that after 60 months, we have a $240 - 60 = 180$-month loan. The payment is still $1,101.09 per month, and the interest rate is still 1 percent per month. The loan balance is thus the present value of the remaining payments:

$$\text{Loan balance} = \$1,101.09 \times [(1 - 1/1.01^{180})/.01]$$
$$= \$1,101.09 \times 83.3217$$
$$= \$91,744.69$$

The balloon payment is a substantial $91,744. Why is it so large? To get an idea, consider the first payment on the mortgage. The interest in the first month is $100,000 \times .01 = \$1,000$. Your payment is $1,101.09, so the loan balance declines by only $101.09. Because the loan balance declines so slowly, the cumulative "pay down" over five years is not great.

We will close this section with an example that may be of particular relevance. Federal Stafford loans are an important source of financing for many college students, helping to cover the cost of tuition, books, new cars, condominiums, and many other things. Sometimes students do not seem to fully realize that Stafford loans have a serious drawback: they must be repaid in monthly installments, usually beginning six months after the student leaves school.

Some Stafford loans are subsidized, meaning that the interest does not begin to accrue until repayment begins (this is a good thing). If you are a dependent undergraduate student under this particular option, the total debt you can run up is, at most, $23,000. The maximum interest rate is 8.25 percent, or $8.25/12 = 0.6875$ percent per month. Under the "standard repayment plan," the loans are amortized over 10 years (subject to a minimum payment of $50).

Suppose you max out borrowing under this program and also get stuck paying the maximum interest rate. Beginning six months after you graduate (or otherwise depart the ivory tower), what will your monthly payment be? How much will you owe after making payments for four years?

Given our earlier discussions, see if you don't agree that your monthly payment assuming a $23,000 total loan is $282.10 per month. Also, as explained in Example 4.28, after making payments for four years, you still owe the present value of the remaining payments. There are 120 payments in all. After you make 48 of them (the first four years), you have 72 to go. By now, it should be easy for you to verify that the present value of $282.10 per month for 72 months at 0.6875 percent per month is just under $16,000, so you still have a long way to go.

Of course, it is possible to rack up much larger debts. According to the Association of American Medical Colleges, medical students who borrowed to attend medical school and graduated in 2008 had an average student loan balance of $154,607. Ouch! How long will it take the average student to pay off her medical school loans?

Loan amortization is a common spreadsheet application. To illustrate, we will set up the problem that we examined earlier: a five-year, $5,000, 9 percent loan with constant payments. Our spreadsheet looks like this:

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| 2 | | | ... | | | | | |

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| 2 | | | **Using a spreadsheet to amortize a loan** | | | | | |
| 3 | | | | | | | | |
| 4 | | | Loan amount: | $5,000 | | | | |
| 5 | | | Interest rate: | 0.09 | | | | |
| 6 | | | Loan term: | 5 | | | | |
| 7 | | | Loan payment: | **$1,285.46** | | | | |
| 8 | | | | Note: Payment is calculated using PMT(rate, nper, -pv, fv). | | | | |
| 9 | | | *Amortization table:* | | | | | |
| 10 | | | | | | | | |
| 11 | | Year | Beginning | Total | Interest | Principal | Ending | |
| 12 | | | Balance | Payment | Paid | Paid | Balance | |
| 13 | | 1 | $5,000.00 | $1,285.46 | $450.00 | $835.46 | $4,164.54 | |
| 14 | | 2 | 4,164.54 | 1,285.46 | 374.81 | 910.65 | 3,253.88 | |
| 15 | | 3 | 3,253.88 | 1,285.46 | 292.85 | 992.61 | 2,261.27 | |
| 16 | | 4 | 2,261.27 | 1,285.46 | 203.51 | 1,081.95 | 1,179.32 | |
| 17 | | 5 | 1,179.32 | 1,285.46 | 106.14 | 1,179.32 | 0.00 | |
| 18 | | Totals | | 6,427.31 | 1,427.31 | 5,000.00 | | |
| 19 | | | | | | | | |
| 20 | | | *Formulas in the amortization table:* | | | | | |
| 21 | | | | | | | | |
| 22 | | Year | Beginning | Total | Interest | Principal | Ending | |
| 23 | | | Balance | Payment | Paid | Paid | Balance | |
| 24 | | 1 | =+D4 | =$D$7 | =+$D$5*C13 | =+D13-E13 | =+C13-F13 | |
| 25 | | 2 | =+G13 | =$D$7 | =+$D$5*C14 | =+D14-E14 | =+C14-F14 | |
| 26 | | 3 | =+G14 | =$D$7 | =+$D$5*C15 | =+D15-E15 | =+C15-F15 | |
| 27 | | 4 | =+G15 | =$D$7 | =+$D$5*C16 | =+D16-E16 | =+C16-F16 | |
| 28 | | 5 | =+G16 | =$D$7 | =+$D$5*C17 | =+D17-E17 | =+C17-F17 | |
| 29 | | | | | | | | |
| 30 | | | Note: Totals in the amortization table are calculated using the SUM formula. | | | | | |
| 31 | | | | | | | | |

Let's say she makes a monthly payment of $1,000, and the loan has an interest rate of 7 percent per year, or .5833 percent per month. See if you agree that it will take 399 months, or just over 33 years, to pay off the loan. Maybe MD really stands for "mucho debt!"

## 4.6   WHAT IS A FIRM WORTH?

Suppose you are in the business of trying to determine the value of small companies. (You are a business appraiser.) How can you determine what a firm is worth? One way to think about the question of how much a firm is worth is to calculate the present value of its future cash flows.

Let us consider the example of a firm that is expected to generate net cash flows (cash inflows minus cash outflows) of $5,000 in the first year and $2,000 for each of the next five years. The firm can be sold for $10,000 seven years from now. The owners of the firm would like to be able to make 10 percent on their investment in the firm.

The value of the firm is found by multiplying the net cash flows by the appropriate present value factor. The value of the firm is simply the sum of the present values of the individual net cash flows.

The present value of the net cash flows is given next.

| | THE PRESENT VALUE OF THE FIRM | | |
|---|---|---|---|
| END OF YEAR | NET CASH FLOW OF THE FIRM | PRESENT VALUE FACTOR (10%) | PRESENT VALUE OF NET CASH FLOWS |
| 1 | $ 5,000 | .90909 | $ 4,545.45 |
| 2 | 2,000 | .82645 | 1,652.90 |
| 3 | 2,000 | .75131 | 1,502.62 |
| 4 | 2,000 | .68301 | 1,366.02 |
| 5 | 2,000 | .62092 | 1,241.84 |
| 6 | 2,000 | .56447 | 1,128.94 |
| 7 | 10,000 | .51316 | 5,131.58 |
| | | Present value of firm | $16,569.35 |

We can also use the simplifying formula for an annuity to give us

$$\frac{\$5,000}{1.1} + \frac{(2{,}000 \times \text{PVIFA}_{10\%,5})}{1.1} + \frac{10{,}000}{(1.1)^7} = \$16{,}569.35$$

Suppose you have the opportunity to acquire the firm for $12,000. Should you acquire the firm? The answer is yes because the NPV is positive.

$$\textbf{NPV} = \textbf{PV} - \textbf{Cost}$$
$$\$4{,}569.35 = \$16{,}569.35 - \$12{,}000$$

The incremental value (NPV) of acquiring the firm is $4,569.35.

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**EXAMPLE 4.29**

**Firm Valuation**

The Trojan Pizza Company is contemplating investing $1 million in four new outlets in Los Angeles. Andrew Lo, the firm's chief financial officer (CFO), has estimated that the investments will pay out cash flows of $200,000 per year for nine years and nothing thereafter. (The cash flows will occur at the end of each year and there will be no cash flow after year 9.) Mr. Lo has determined that the relevant discount rate for this investment is 15 percent. This is the rate of return that the firm can earn at comparable projects. Should the Trojan Pizza Company make the investments in the new outlets?

The decision can be evaluated as:

$$\text{NPV} = -\$1{,}000{,}000 + \frac{\$200{,}000}{1.15} + \frac{\$200{,}000}{(1.15)^2} + \cdots + \frac{\$200{,}000}{(1.15)^9}$$
$$= -\$1{,}000{,}000 + \$200{,}000 \times \text{PVIFA}_{15\%,9}$$
$$= -\$1{,}000{,}000 + \$954{,}316.78$$
$$= -\$45{,}683.22$$

The present value of the four new outlets is only $954,316.78. The outlets are worth less than they cost. The Trojan Pizza Company should not make the investment because the NPV is −$45,683.22. If the Trojan Pizza Company requires a 15 percent rate of return, the new outlets are not a good investment.

## SUMMARY AND CONCLUSIONS

1. Two basic concepts, *future value* and *present value*, were introduced in the beginning of this chapter. With a 10 percent interest rate, an investor with $1 today can generate a future value of $1.10 in a year, $1.21 [$1 × (1.10)$^2$] in two years, and so on. Conversely, present value analysis places a current value on a later cash flow. With the same 10 percent interest rate, a dollar to be received in one year has a present value of $0.909 ($1/1.10) in year 0. A dollar to be received in two years has a present value of $0.826 [$1/(1.10)$^2$].

2. One commonly expresses the interest rate as, say, 12 percent per year. However, one can speak of the interest rate as 3 percent per quarter. Although the stated annual interest rate remains 12 percent (3 percent × 4), the effective annual interest rate is 12.55 percent [(1.03)$^4$ − 1]. In other words, the compounding process increases the future value of an investment. The limiting case is continuous compounding, where funds are assumed to be reinvested every infinitesimal instant.

3. A basic quantitative technique for financial decision making is net present value analysis. The net present value formula for an investment that generates cash flows ($C_i$) in future periods is

$$\text{NPV} = -C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = -C_0 + \sum_{i=1}^{T} \frac{C_i}{(1+r)^i}$$

The formula assumes that the cash flow at date 0 is the initial investment (a cash outflow).

4. Frequently, the actual calculation of present value is long and tedious. The computation of the present value of a long-term mortgage with monthly payments is a good example of this. We presented four simplifying formulas:

$$\textbf{Perpetuity: } \text{PV} = \frac{C}{r}$$

$$\textbf{Growing perpetuity: } \text{PV} = \frac{C}{r-g}$$

$$\textbf{Annuity: } \text{PV} = C\left[\frac{1 - \frac{1}{(1+r)^T}}{r}\right]$$

$$\textbf{Growing annuity: } \text{PV} = C\left[\frac{1 - \left(\frac{1+g}{1+r}\right)^T}{r-g}\right]$$

5. We stressed a few practical considerations in the application of these formulas:

   a. The numerator in each of the formulas, $C$, is the cash flow to be received *one full period hence*.

   b. Cash flows are generally irregular in practice. To avoid unwieldy problems, assumptions to create more regular cash flows are made both in this textbook and in the real world.

   c. A number of present value problems involve annuities (or perpetuities) beginning a few periods hence. Students should practice combining the annuity (or perpetuity) formula with the discounting formula to solve these problems.

   d. Annuities and perpetuities may have periods of every two or every *n* years, rather than once a year. The annuity and perpetuity formulas can easily handle such circumstances.

   e. One frequently encounters problems where the present value of one annuity must be equated with the present value of another annuity.

6. Many loans are annuities. The process of providing for a loan to be paid off gradually is called amortizing the loan, and we discussed how amortization schedules are prepared and interpreted.

## CONCEPT QUESTIONS

1. **Compounding and Period**    As you increase the length of time involved, what happens to future values? What happens to present values?

2. **Interest Rates**    What happens to the future value of an annuity if you increase the rate $r$? What happens to the present value?

3. **Present Value**    Suppose two athletes sign 10-year contracts for $80 million. In one case, we're told that the $80 million will be paid in 10 equal installments. In the other case, we're told that the $80 million will be paid in 10 installments, but the installments will increase by 5 percent per year. Who got the better deal?

4. **APR and EAR**    Should lending laws be changed to require lenders to report EARs instead of APRs? Why or why not?

5. **Time Value**    On subsidized Stafford loans, a common source of financial aid for college students, interest does not begin to accrue until repayment begins. Who receives a bigger subsidy, a freshman or a senior? Explain.

Use the following information for Questions 6–10.

Toyota Motor Credit Corporation (TMCC), a subsidiary of Toyota Motor Corporation, offered some securities for sale to the public on March 28, 2008. Under the terms of the deal, TMCC promised to repay the owner of one of these securities $100,000 on March 28, 2038, but investors would receive nothing until then. Investors paid TMCC $24,099 for each of these securities, so they gave up $24,099 on March 28, 2008, for the promise of a $100,000 payment 30 years later.

6. **Time Value of Money**    Why would TMCC be willing to accept such a small amount today ($24,099) in exchange for a promise to repay about four times that amount ($100,000) in the future?

7. **Call Provisions**    TMCC has the right to buy back the securities on the anniversary date at a price established when the securities were issued (this feature is a term of this particular deal). What impact does this feature have on the desirability of this security as an investment?

8. **Time Value of Money**    Would you be willing to pay $24,099 today in exchange for $100,000 in 30 years? What would be the key considerations in answering yes or no? Would your answer depend on who is making the promise to repay?

9. **Investment Comparison**    Suppose that when TMCC offered the security for $24,099 the U.S. Treasury had offered an essentially identical security. Do you think it would have had a higher or lower price? Why?

10. **Length of Investment**    The TMCC security is bought and sold on the New York Stock Exchange. If you looked at the price today, do you think the price would exceed the $24,099 original price? Why? If you looked in the year 2019, do you think the price would be higher or lower than today's price? Why?

## QUESTIONS AND PROBLEMS

**Connect**

**Basic**
(Questions 1–20)

1. **Simple Interest versus Compound Interest**    First City Bank pays 7 percent simple interest on its savings account balances, whereas Second City Bank pays 7 percent interest compounded annually. If you made a $6,000 deposit in each bank, how much more money would you earn from your Second City Bank account at the end of 10 years?

2. **Calculating Future Values**    Compute the future value of $2,500 compounded annually for

   a. 10 years at 6 percent

   b. 10 years at 8 percent

**c.** 20 years at 6 percent

**d.** Why is the interest earned in part (c) not twice the amount earned in part (a)?

3. **Calculating Present Values**  For each of the following, compute the present value:

| PRESENT VALUE | YEARS | INTEREST RATE | FUTURE VALUE |
|---|---|---|---|
| | 9 | 7% | $  15,451 |
| | 6 | 9 | 51,557 |
| | 21 | 14 | 886,073 |
| | 27 | 16 | 550,164 |

4. **Calculating Interest Rates**  Solve for the unknown interest rate in each of the following:

| PRESENT VALUE | YEARS | INTEREST RATE | FUTURE VALUE |
|---|---|---|---|
| $    243 | 3 | | $    307 |
| 405 | 10 | | 896 |
| 34,500 | 13 | | 162,181 |
| 51,285 | 26 | | 483,500 |

5. **Calculating the Number of Periods**  Solve for the unknown number of years in each of the following:

| PRESENT VALUE | YEARS | INTEREST RATE | FUTURE VALUE |
|---|---|---|---|
| $    625 | | 7% | $  1,284 |
| 810 | | 8 | 4,341 |
| 18,400 | | 13 | 402,662 |
| 21,500 | | 16 | 173,439 |

6. **Calculating the Number of Periods**  At 8 percent interest, how long does it take to double your money? To quadruple it?

7. **Calculating Present Values**  Imprudential, Inc., has an unfunded pension liability of $750 million that must be paid in 20 years. To assess the value of the firm's stock, financial analysts want to discount this liability back to the present. If the relevant discount rate is 6.25 percent, what is the present value of this liability?

8. **Calculating Rates of Return**  Although appealing to more refined tastes, art as a collectible has not always performed so profitably. During 2003, Sothebys sold the Edgar Degas bronze sculpture *Petite Danseuse de Quatorze Ans* at auction for a price of $10,311,500. Unfortunately for the previous owner, he had purchased it in 1999 at a price of $12,377,500. What was his annual rate of return on this sculpture?

9. **Perpetuities**  An investor purchasing a British consol is entitled to receive annual payments from the British government forever. What is the price of a consol that pays $160 annually if the next payment occurs one year from today? The market interest rate is 4.5 percent.

**10. Continuous Compounding**    Compute the future value of $1,800 continuously compounded for

    **a.**   Five years at a stated annual interest rate of 14 percent.

    **b.**   Three years at a stated annual interest rate of 6 percent.

    **c.**   Ten years at a stated annual interest rate of 7 percent.

    **d.**   Eight years at a stated annual interest rate of 9 percent.

**11. Present Value and Multiple Cash Flows**    Conoly Co. has identified an investment project with the following cash flows. If the discount rate is 5 percent, what is the present value of these cash flows? What is the present value at 13 percent? At 18 percent?

| YEAR | CASH FLOW |
|------|-----------|
| 1 | $  850 |
| 2 | 740 |
| 3 | 1,090 |
| 4 | 1,310 |

**12. Present Value and Multiple Cash Flows**    Investment X offers to pay you $6,000 per year for nine years, whereas Investment Y offers to pay you $8,500 per year for five years. Which of these cash flow streams has the higher present value if the discount rate is 9 percent? If the discount rate is 21 percent?

**13. Calculating Annuity Present Value**    An investment offers $7,000 per year for 15 years, with the first payment occurring one year from now. If the required return is 8 percent, what is the value of the investment? What would the value be if the payments occurred for 40 years? For 75 years? Forever?

**14. Calculating Perpetuity Values**    The Perpetual Life Insurance Co. is trying to sell you an investment policy that will pay you and your heirs $25,000 per year forever. If the required return on this investment is 6 percent, how much will you pay for the policy? Suppose the Perpetual Life Insurance Co. told you the policy costs $435,000. At what interest rate would this be a fair deal?

**15. Calculating EAR**    Find the EAR in each of the following cases:

| STATED RATE (APR) | NUMBER OF TIMES COMPOUNDED | EFFECTIVE RATE (EAR) |
|-------------------|----------------------------|----------------------|
| 15% | Quarterly | |
| 12 | Monthly | |
| 9 | Daily | |
| 13 | Infinite | |

**16. Calculating APR**    Find the APR, or stated rate, in each of the following cases:

| STATED RATE (APR) | NUMBER OF TIMES COMPOUNDED | EFFECTIVE RATE (EAR) |
|-------------------|----------------------------|----------------------|
| | Semiannually | 10.2% |
| | Monthly | 8.4 |
| | Weekly | 15.9 |
| | Infinite | 18.7 |

17. **Calculating EAR** First National Bank charges 15.1 percent compounded monthly on its business loans. First United Bank charges 15.5 percent compounded semiannually. As a potential borrower, which bank would you go to for a new loan?

18. **Interest Rates** Well-known financial writer Andrew Tobias argues that he can earn 177 percent per year buying wine by the case. Specifically, he assumes that he will consume one $10 bottle of fine Bordeaux per week for the next twelve weeks. He can either pay $10 per week or buy a case of 12 bottles today. If he buys the case, he receives a 10 percent discount, and, by doing so, earns the 177 percent. Assume he buys the wine and consumes the first bottle today. Do you agree with his analysis? Do you see a problem with his numbers?

19. **Calculating Number of Periods** One of your customers is delinquent on his accounts payable balance. You've mutually agreed to a repayment schedule of $375 per month. You will charge 0.9 percent per month interest on the overdue balance. If the current balance is $13,200, how long will it take for the account to be paid off?

20. **Calculating EAR** Friendly's Quick Loans, Inc., offers you "three for four or I knock on your door." This means you get $3 today and repay $4 when you get your paycheck in one week (or else). What's the effective annual return Friendly's earns on this lending business? If you were brave enough to ask, what APR would Friendly's say you were paying?

21. **Future Value** What is the future value in three years of $1,800 invested in an account with a stated annual interest rate of 10 percent,

   **a.** Compounded annually?

   **b.** Compounded semiannually?

   **c.** Compounded monthly?

   **d.** Compounded continuously?

   **e.** Why does the future value increase as the compounding period shortens?

22. **Simple Interest versus Compound Interest** First Simple Bank pays 7 percent simple interest on its investment accounts. If First Complex Bank pays interest on its accounts compounded annually, what rate should the bank set if it wants to match First Simple Bank over an investment horizon of 10 years?

23. **Calculating Annuities** You are planning to save for retirement over the next 30 years. To do this, you will invest $700 a month in a stock account and $300 a month in a bond account. The return of the stock account is expected to be 11 percent, and the bond account will pay 6 percent. When you retire, you will combine your money into an account with an 8 percent return. How much can you withdraw each month from your account assuming a 25-year withdrawal period?

24. **Calculating Rates of Return** Suppose an investment offers to quintuple your money in 12 months (don't believe it). What rate of return per quarter are you being offered?

25. **Calculating Rates of Return** You're trying to choose between two different investments, both of which have up-front costs of $75,000. Investment G returns $125,000 in five years. Investment H returns $245,000 in 11 years. Which of these investments has the higher return?

26. **Growing Perpetuities** Mark Weinstein has been working on an advanced technology in laser eye surgery. His technology will be available in the near term. He anticipates his first annual cash flow from the technology to be $210,000, received three years from today. Subsequent annual cash flows will grow at 3 percent, in perpetuity. What is the present value of the technology if the discount rate is 12 percent?

27. **Perpetuities** A prestigious investment bank designed a new security that pays a quarterly dividend of $3 in perpetuity. The first dividend occurs one quarter from today. What is the price of the security if the stated annual interest rate is 9 percent, compounded quarterly?

**Intermediate**
(Questions 21–52)

28. **Annuity Present Values**    What is the present value of an annuity of $6,000 per year, with the first cash flow received four years from today and the last one received 18 years from today? Use a discount rate of 8 percent.

29. **Annuity Present Values**    What is the value today of a 15-year annuity that pays $750 a year? The annuity's first payment occurs six years from today. The annual interest rate is 9 percent for years 1 through 5, and 12 percent thereafter.

30. **Balloon Payments**    Mike Bayles has just arranged to purchase a $750,000 vacation home in the Bahamas with a 25 percent down payment. The mortgage has a 6.5 percent stated annual interest rate, compounded monthly, and calls for equal monthly payments over the next 30 years. His first payment will be due one month from now. However, the mortgage has an eight-year balloon payment, meaning that the balance of the loan must be paid off at the end of year 8. There were no other transaction costs or finance charges. How much will Mike's balloon payment be in eight years?

31. **Calculating Interest Expense**    You receive a credit card application from Shady Banks Savings and Loan offering an introductory rate of 1.80 percent per year, compounded monthly for the first six months, increasing thereafter to 18 percent compounded monthly. Assuming you transfer the $6,000 balance from your existing credit card and make no subsequent payments, how much interest will you owe at the end of the first year?

32. **Perpetuities**    Barrett Pharmaceuticals is considering a drug project that costs $875,000 today and is expected to generate end-of-year annual cash flows of $61,000, forever. At what discount rate would Barrett be indifferent between accepting or rejecting the project?

33. **Growing Annuity**    Southern California Publishing Company is trying to decide whether or not to revise its popular textbook, *Financial Psychoanalysis Made Simple*. It has estimated that the revision will cost $95,000. Cash flows from increased sales will be $26,000 the first year. These cash flows will increase by 6 percent per year. The book will go out of print five years from now. Assume that the initial cost is paid now and revenues are received at the end of each year. If the company requires an 11 percent return for such an investment, should it undertake the revision?

34. **Growing Annuity**    Your job pays you only once a year, for all the work you did over the previous 12 months. Today, December 31, you just received your salary of $75,000 and you plan to spend all of it. However, you want to start saving for retirement beginning next year. You have decided that one year from today you will begin depositing 10 percent of your annual salary in an account that will earn 9 percent per year. Your salary will increase at 4 percent per year throughout your career. How much money will you have on the date of your retirement 35 years from today?

35. **Present Value and Interest Rates**    What is the relationship between the value of an annuity and the level of interest rates? Suppose you just bought a 12-year annuity of $7,000 per year at the current interest rate of 10 percent per year. What happens to the value of your investment if interest rates suddenly drop to 5 percent? What if interest rates suddenly rise to 15 percent?

36. **Calculating the Number of Payments**    You're prepared to make monthly payments of $125, beginning at the end of this month, into an account that pays 10 percent interest compounded monthly. How many payments will you have made when your account balance reaches $25,000?

37. **Calculating Annuity Present Values**    You want to borrow $75,000 from your local bank to buy a new sailboat. You can afford to make monthly payments of $1,475, but no more. Assuming monthly compounding, what is the highest rate you can afford on a 60-month APR loan?

38. **Calculating Loan Payments**    You need a 30-year, fixed-rate mortgage to buy a new home for $260,000. Your mortgage bank will lend you the money at a 6.1 percent APR for this loan. However, you can only afford monthly payments of $1,150, so you offer to pay off any remaining loan balance at the end of the loan in the form of a single balloon payment. How large will this balloon payment have to be for you to keep your monthly payments at $1,150?

39. **Present and Future Values**    The present value of the following cash flow stream is $5,985 when discounted at 10 percent annually. What is the value of the missing cash flow?

| YEAR | CASH FLOW |
|------|-----------|
| 1 | $1,750 |
| 2 | ? |
| 3 | 1,380 |
| 4 | 2,230 |

40. **Calculating Present Values**    You just won the TVM Lottery. You will receive $1 million today plus another 10 annual payments that increase by $350,000 per year. Thus, in one year you receive $1.35 million. In two years, you get $1.7 million, and so on. If the appropriate interest rate is 8 percent, what is the present value of your winnings?

41. **EAR versus APR**    You have just purchased a new warehouse. To finance the purchase, you've arranged for a 30-year mortgage loan for 80 percent of the $2,400,000 purchase price. The monthly payment on this loan will be $13,500. What is the APR on this loan? The EAR?

42. **Present Value and Break-Even Interest**    Consider a firm with a contract to sell an asset for $140,000 three years from now. The asset costs $91,000 to produce today. Given a relevant discount rate on this asset of 13 percent per year, will the firm make a profit on this asset? At what rate does the firm just break even?

43. **Present Value and Multiple Cash Flows**    What is the present value of $2,500 per year, at a discount rate of 8 percent, if the first payment is received 7 years from now and the last payment is received 30 years from now?

44. **Variable Interest Rates**    A 15-year annuity pays $1,700 per month, and payments are made at the end of each month. If the interest rate is 12 percent compounded monthly for the first seven years, and 9 percent compounded monthly thereafter, what is the present value of the annuity?

45. **Comparing Cash Flow Streams**    You have your choice of two investment accounts. Investment A is a 15-year annuity that features end-of-month $1,300 payments and has an interest rate of 8.75 percent compounded monthly. Investment B is an 8 percent continuously compounded lump-sum investment, also good for 15 years. How much money would you need to invest in B today for it to be worth as much as Investment A 15 years from now?

46. **Calculating Present Value of a Perpetuity**    Given an interest rate of 8.2 percent per year, what is the value at date $t = 7$ of a perpetual stream of $2,100 payments that begin at date $t = 15$?

47. **Calculating EAR**    A local finance company quotes a 17 percent interest rate on one-year loans. So, if you borrow $15,000, the interest for the year will be $2,550. Because you must repay a total of $17,550 in one year, the finance company requires you to pay $17,550/12, or $1,462.50, per month

over the next 12 months. Is this a 17 percent loan? What rate would legally have to be quoted? What is the effective annual rate?

**48. Calculating Present Values** A 5-year annuity of ten $10,000 semiannual payments will begin 9 years from now, with the first payment coming 9.5 years from now. If the discount rate is 10 percent compounded monthly, what is the value of this annuity five years from now? What is the value three years from now? What is the current value of the annuity?

**49. Calculating Annuities Due** Suppose you are going to receive $8,000 per year for 10 years. The appropriate interest rate is 9 percent.

    **a.** What is the present value of the payments if they are in the form of an ordinary annuity? What is the present value if the payments are an annuity due?

    **b.** Suppose you plan to invest the payments for 10 years. What is the future value if the payments are an ordinary annuity? What if the payments are an annuity due?

    **c.** Which has the highest present value, the ordinary annuity or the annuity due? Which has the highest future value? Will this always be true?

**50. Calculating Annuities Due** You want to buy a new sports car from Muscle Motors for $85,000. The contract is in the form of a 60-month annuity due at a 6.8 percent APR. What will your monthly payment be?

**51. Amortization with Equal Payments** Prepare an amortization schedule for a three-year loan of $69,000. The interest rate is 9 percent per year, and the loan calls for equal annual payments. How much interest is paid in the third year? How much total interest is paid over the life of the loan?

**52. Amortization with Equal Principal Payments** Rework Problem 51 assuming that the loan agreement calls for a principal reduction of $23,000 every year instead of equal annual payments.

**Challenge**
(Questions 53–80)

**53. Calculating Annuities Due** You want to lease a set of golf clubs from Pings Ltd. The lease contract is in the form of 24 equal monthly payments at an 11.50 percent stated annual interest rate, compounded monthly. Since the clubs cost $3,500 retail, Pings wants the PV of the lease payments to equal $3,500. Suppose that your first payment is due immediately. What will your monthly lease payments be?

**54. Annuities** You are saving for the college education of your two children. They are two years apart in age; one will begin college 15 years from today and the other will begin 17 years from today. You estimate your children's college expenses to be $55,000 per year per child, payable at the beginning of each school year. The annual interest rate is 7.25 percent. How much money must you deposit in an account each year to fund your children's education? Your deposits begin one year from today. You will make your last deposit when your oldest child enters college. Assume four years of college.

**55. Growing Annuities** Tom Adams has received a job offer from a large investment bank as a clerk to an associate banker. His base salary will be $52,000. He will receive his first annual salary payment one year from the day he begins to work. In addition, he will get an immediate $10,000 bonus for joining the company. His salary will grow at 3.5 percent each year. Each year he will receive a bonus equal to 10 percent of his salary. Mr. Adams is expected to work for 35 years. What is the present value of the offer if the discount rate is 9 percent?

**56. Calculating Annuities** You have recently won the super jackpot in the Set for Life lottery. On reading the fine print, you discover that you have the following two options:

    **a.** You will receive 31 annual payments of $400,000, with the first payment being delivered today. The income will be taxed at a rate of 35 percent. Taxes will be withheld when the checks are issued.

**b.** You will receive $900,000 now, and you will not have to pay taxes on this amount. In addition, beginning one year from today, you will receive $290,000 each year for 30 years. The cash flows from this annuity will be taxed at 35 percent.

Using a discount rate of 10 percent, which option should you select?

**57. Calculating Growing Annuities**    You have 30 years left until retirement and want to retire with $2.2 million. Your salary is paid annually and you will receive $80,000 at the end of the current year. Your salary will increase at 3 percent per year, and you can earn a 10 percent return on the money you invest. If you save a constant percentage of your salary, what percentage of your salary must you save each year?

**58. Balloon Payments**    On September 1, 2008, Susan Chao bought a motorcycle for $30,000. She paid $1,000 down and financed the balance with a five-year loan at a stated annual interest rate of 7.8 percent, compounded monthly. She started the monthly payments exactly one month after the purchase (i.e., October 1, 2008). Two years later, at the end of October 2010, Susan got a new job and decided to pay off the loan. If the bank charges her a 1 percent prepayment penalty based on the loan balance, how much must she pay the bank on November 1, 2010?

**59. Calculating Annuity Values**    Bilbo Baggins wants to save money to meet three objectives. First, he would like to be able to retire 30 years from now with a retirement income of $15,000 per month for 20 years, with the first payment received 30 years and 1 month from now. Second, he would like to purchase a cabin in Rivendell in 10 years at an estimated cost of $300,000. Third, after he passes on at the end of the 20 years of withdrawals, he would like to leave an inheritance of $1,000,000 to his nephew Frodo. He can afford to save $2,000 per month for the next 10 years. If he can earn a 10 percent EAR before he retires and an 8 percent EAR after he retires, how much will he have to save each month in years 11 through 30?

**60. Calculating Annuity Values**    After deciding to buy a new car, you can either lease the car or purchase it with a 3-year loan. The car you wish to buy costs $30,000. The dealer has a special leasing arrangement where you pay $1,500 today and $450 per month for the next three years. If you purchase the car, you will pay it off in monthly payments over the next three years at an 8 percent APR. You believe that you will be able to sell the car for $19,000 in three years. Should you buy or lease the car? What break-even resale price in three years would make you indifferent between buying and leasing?

**61. Calculating Annuity Values**    An All-Pro defensive lineman is in contract negotiations. The team has offered the following salary structure:

| TIME | SALARY |
|------|--------|
| 0 | $5,000,000 |
| 1 | $4,000,000 |
| 2 | $4,800,000 |
| 3 | $5,600,000 |
| 4 | $6,200,000 |
| 5 | $6,800,000 |
| 6 | $7,300,000 |

All salaries are to be paid in a lump sum. The player has asked you as his agent to renegotiate the terms. He wants an $8 million signing bonus payable today and a contract value increase of $1,500,000. He also wants an equal salary paid every three months, with the first paycheck three

months from now. If the interest rate is 5 percent compounded daily, what is the amount of his quarterly check? Assume 365 days in a year.

62. **Discount Interest Loans** This question illustrates what is known as *discount interest*. Imagine you are discussing a loan with a somewhat unscrupulous lender. You want to borrow $20,000 for one year. The interest rate is 14 percent. You and the lender agree that the interest on the loan will be .14 × $20,000 = $2,800. So the lender deducts this interest amount from the loan up front and gives you $17,200. In this case, we say that the discount is $2,800. What's wrong here?

63. **Calculating Annuity Values** You are serving on a jury. A plaintiff is suing the city for injuries sustained after a freak street sweeper accident. In the trial, doctors testified that it will be five years before the plaintiff is able to return to work. The jury has already decided in favor of the plaintiff. You are the foreperson of the jury and propose that the jury give the plaintiff an award to cover the following: 1) The present value of two years' back pay. The plaintiff's annual salary for the last two years would have been $38,000 and $40,000, respectively. 2) The present value of five years' future salary. You assume the salary will be $45,000 per year. 3) $200,000 for pain and suffering. 4) $30,000 for court costs. Assume that the salary payments are equal amounts paid at the end of each month. If the interest rate you choose is a 7 percent EAR, what is the size of the settlement? If you were the plaintiff, would you like to see a higher or lower interest rate?

64. **Calculating EAR with Points** You are looking at a one-year loan of $10,000. The interest rate is quoted as 9 percent plus two points. A *point* on a loan is simply 1 percent (one percentage point) of the loan amount. Quotes similar to this one are very common with home mortgages. The interest rate quotation in this example requires the borrower to pay two points to the lender up front and repay the loan later with 9 percent interest. What rate would you actually be paying here?

65. **Calculating EAR with Points** The interest rate on a one-year loan is quoted as 13 percent plus three points (see the previous problem). What is the EAR? Is your answer affected by the loan amount?

66. **EAR versus APR** There are two banks in the area that offer 30-year, $225,000 mortgages at 7.5 percent and charge a $2,500 loan application fee. However, the application fee charged by Insecurity Bank and Trust is refundable if the loan application is denied, whereas that charged by I. M. Greedy and Sons Mortgage Bank is not. The current disclosure law requires that any fees that will be refunded if the applicant is rejected be included in calculating the APR, but this is not required with nonrefundable fees (presumably because refundable fees are part of the loan rather than a fee). What are the EARs on these two loans? What are the APRs?

67. **Calculating EAR with Add-On Interest** This problem illustrates a deceptive way of quoting interest rates called *add-on interest*. Imagine that you see an advertisement for Crazy Judy's Stereo City that reads something like this: "$2,000 Instant Credit! 17% Simple Interest! Three Years to Pay! Low, Low Monthly Payments!" You're not exactly sure what all this means and somebody has spilled ink over the APR on the loan contract, so you ask the manager for clarification.

Judy explains that if you borrow $2,000 for three years at 17 percent interest, in three years you will owe:

$$\$2,000 \times 1.17^3 = \$2,000 \times 1.601613 = \$3,203.23$$

Now, Judy recognizes that coming up with $3,203.23 all at once might be a strain, so she lets you make "low, low monthly payments" of $3,203.23/36 = $88.98 per month, even though this is extra bookkeeping work for her.

Is this a 17 percent loan? Why or why not? What is the APR on this loan? What is the EAR? Why do you think this is called add-on interest?

68. **Calculating Annuity Payments**    This is a classic retirement problem. A time line will help in solving it. Your friend is celebrating her 35th birthday today and wants to start saving for her anticipated retirement at age 65. She wants to be able to withdraw $140,000 from her savings account on each birthday for 20 years following her retirement; the first withdrawal will be on her 66th birthday. Your friend intends to invest her money in the local credit union, which offers 7 percent interest per year. She wants to make equal annual payments on each birthday into the account established at the credit union for her retirement fund.

   a. If she starts making these deposits on her 36th birthday and continues to make deposits until she is 65 (the last deposit will be on her 65th birthday), what amount must she deposit annually to be able to make the desired withdrawals at retirement?

   b. Suppose your friend has just inherited a large sum of money. Rather than making equal annual payments, she has decided to make one lump-sum payment on her 35th birthday to cover her retirement needs. What amount does she have to deposit?

   c. Suppose your friend's employer will contribute $2,000 to the account every year as part of the company's profit-sharing plan. In addition, your friend expects a $50,000 distribution from a family trust fund on her 55th birthday, which she will also put into the retirement account. What amount must she deposit annually now to be able to make the desired withdrawals at retirement?

69. **Calculating the Number of Periods**    Your Christmas ski vacation was great, but it unfortunately ran a bit over budget. All is not lost, because you just received an offer in the mail to transfer your $10,000 balance from your current credit card, which charges an annual rate of 19.2 percent, to a new credit card charging a rate of 9.2 percent. How much faster could you pay the loan off by making your planned monthly payments of $170 with the new card? What if there was a 3 percent fee charged on any balances transferred?

70. **Future Value and Multiple Cash Flows**    An insurance company is offering a new policy to its customers. Typically, the policy is bought by a parent or grandparent for a child at the child's birth. The details of the policy are as follows: The purchaser (say, the parent) makes the following six payments to the insurance company:

   | | |
   |---|---|
   | First birthday: | $ 700 |
   | Second birthday: | $ 700 |
   | Third birthday: | $ 800 |
   | Fourth birthday: | $ 800 |
   | Fifth birthday: | $ 900 |
   | Sixth birthday: | $ 900 |

   After the child's sixth birthday, no more payments are made. When the child reaches age 65, he or she receives $500,000. If the relevant interest rate is 10 percent for the first six years and 8 percent for all subsequent years, is the policy worth buying?

71. **Annuity Present Values and Effective Rates**    You have just won the lottery. You will receive $4,000,000 today, and then receive 40 payments of $1,000,000. These payments will start one year from now and will be paid every six months. A representative from Greenleaf Investments has offered to purchase all the payments from you for $20.4 million. If the appropriate interest rate is an 8 percent APR compounded daily, should you take the offer? Assume there are 365 days per year.

72. **Calculating Interest Rates**    A financial planning service offers a college savings program. The plan calls for you to make six annual payments of $14,000 each, with the first payment occurring

today, your child's 12th birthday. Beginning on your child's 18th birthday, the plan will provide $30,000 per year for four years. What return is this investment offering?

73. **Break-Even Investment Returns**    Your financial planner offers you two different investment plans. Plan X is a $10,000 annual perpetuity. Plan Y is a 10-year, $21,000 annual annuity. Both plans will make their first payment one year from today. At what discount rate would you be indifferent between these two plans?

74. **Perpetual Cash Flows**    What is the value of an investment that pays $17,000 every *other* year forever, if the first payment occurs one year from today and the discount rate is 12 percent compounded daily? What is the value today if the first payment occurs four years from today?

75. **Ordinary Annuities and Annuities Due**    As discussed in the text, an annuity due is identical to an ordinary annuity except that the periodic payments occur at the beginning of each period and not at the end of the period. Show that the relationship between the value of an ordinary annuity and the value of an otherwise equivalent annuity due is:

Annuity due value $=$ Ordinary annuity value $\times (1 + r)$

Show this for both present and future values.

76. **Calculating Annuities Due**    A 10-year annual annuity due with the first payment occurring at date $t = 7$ has a current value of $85,000. If the discount rate is 9 percent per year, what is the annuity payment amount?

77. **Calculating EAR**    A check-cashing store is in the business of making personal loans to walk-up customers. The store makes only one-week loans at 7 percent interest per week.

   a. What APR must the store report to its customers? What is the EAR that the customers are actually paying?

   b. Now suppose the store makes one-week loans at 7 percent discount interest per week (see Question 62). What's the APR now? The EAR?

   c. The check-cashing store also makes one-month add-on interest loans at 7 percent discount interest per week. Thus, if you borrow $100 for one month (four weeks), the interest will be ($100 $\times$ 1.07⁴) $-$ 100 = $31.08. Because this is discount interest, your net loan proceeds today will be $68.92. You must then repay the store $100 at the end of the month. To help you out, though, the store lets you pay off this $100 in installments of $25 per week. What is the APR of this loan? What is the EAR?

78. **Present Value of a Growing Perpetuity**    What is the equation for the present value of a growing perpetuity with a payment of $C$ one period from today if the payments grow by $C$ each period?

79. **Rule of 72**    A useful rule of thumb for the time it takes an investment to double with discrete compounding is the "Rule of 72." To use the Rule of 72, you simply divide 72 by the interest rate to determine the number of periods it takes for a value today to double. For example, if the interest rate is 6 percent, the Rule of 72 says, it will take 72/6 = 12 years to double. This is approximately equal to the actual answer of 11.90 years. The Rule of 72 can also be applied to determine what interest rate is needed to double money in a specified period. This is a useful approximation for many interest rates and periods. At what rate is the Rule of 72 exact?

80. **Rule of 69.3**    A corollary to the Rule of 72 is the Rule of 69.3. The Rule of 69.3 is exactly correct except for rounding when interest rates are compounded continuously. Prove the Rule of 69.3 for continuously compounded interest.

1. **Calculating Future Values** Go to www.dinkytown.net and follow the "Savings Calculator" link. If you currently have $10,000 and invest this money at 9 percent, how much will you have in 30 years? Assume you will not make any additional contributions. How much will you have if you can earn 11 percent?

2. **Calculating the Number of Periods** Go to www.dinkytown.net and follow the "Cool Million" link. You want to be a millionaire. You can earn 11.5 percent per year. Using your current age, at what age will you become a millionaire if you have $25,000 to invest, assuming you make no other deposits (and assuming inflation is zero)?

3. **Future Values and Taxes** Taxes can greatly affect the future value of your investment. The Financial Calculators Web site at www.fincalc.com has a financial calculator that adjusts your return for taxes. Suppose you have $50,000 to invest today. If you can earn a 12 percent return and no additional annual savings, how much will you have in 20 years? (Enter 0 percent as the tax rate.) Now, assume that your marginal tax rate is 27.5 percent. How much will you have at this tax rate?

## THE MBA DECISION

Ben Bates graduated from college six years ago with a finance undergraduate degree. Since graduation, he has been employed in the finance department at East Coast Yachts. Although he is satisfied with his current job, his goal is to become an investment banker. He feels that an MBA degree would allow him to achieve this goal. After examining schools, he has narrowed his choice to either Wilton University or Mount Perry College. Although internships are encouraged by both schools, to get class credit for the internship, no salary can be paid. Other than internships, neither schools will allow its students to work while enrolled in its MBA program.

Ben's annual salary at East Coast Yachts is $50,000 per year, and his salary is expected to increase at 3 percent per year until retirement. He is currently 28 years old and expects to work for 40 more years. His current job includes a fully paid health insurance plan, and his current average tax rate is 26 percent. Ben has a savings account with enough money to cover the entire cost of his MBA program.

The Ritter College of Business at Wilton University is one of the top MBA programs in the country. The MBA degree requires two years of full-time enrollment at the university. The annual tuition is $65,000, payable at the beginning of each school year. Books and other supplies are estimated to cost $2,500 per year. Ben expects that after graduation from Wilton, he will receive a job offer for about $90,000 per year, with a $15,000 signing bonus. The salary at this job will increase at 4 percent per year. Because of the higher salary, his average income tax rate will increase to 31 percent.

The Bradley School of Business at Mount Perry College began its MBA program 16 years ago. The Bradley School is smaller and less well known than the Ritter College. Bradley offers an accelerated, one-year program, with a tuition cost of $75,000 to be paid upon matriculation. Books and other supplies for the program are expected to cost $3,500. Ben thinks that after graduation from Mount Perry, he will receive an offer of $78,000 per year, with a $12,000 signing bonus. The salary at this job will increase at 3.5 percent per year. His average income tax rate at this level of income will be 29 percent.

Both schools offer a health insurance plan that will cost $3,000 per year, payable at the beginning of the year. Ben also estimates that room and board expenses will cost $2,000 more per year at both

schools than his current expenses, payable at the beginning of each year. The appropriate discount rate is 6.5 percent. Assume all salaries are paid at the end of each year.

1. How does Ben's age affect his decision to get an MBA?

2. What other, perhaps nonquantifiable factors, affect Ben's decision to get an MBA?

3. Assuming all salaries are paid at the end of each year, what is the best option for Ben—from a strictly financial standpoint?

4. In choosing between the two schools, Ben believes that the appropriate analysis is to calculate the future value of each option. How would you evaluate this statement?

5. What initial salary would Ben need to receive to make him indifferent between attending Wilton University and staying in his current position? Assume his tax rate after graduating from Wilton University will be 31 percent regardless of his income level.

6. Suppose that instead of being able to pay cash for his MBA, Ben must borrow the money. The current borrowing rate is 5.4 percent. How would this affect his decision to get an MBA?